Functional Analysis and Applications IV Prerequisites Review

(Or What Do I Need to Know for Functional Analysis?)

MATH 4371/42920

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These notes contain definitions and theorem which you should have seen in modules which are prerequisite for Functional Analysis and Applications IV. You are required to know (and sometimes use) the content of these notes, you will not be asked to prove them in our module.

1. Zorn's lemma

DEFINITION. A partially ordered set is a set M on which there exists a binary operation, denoted by $\leq$, which satisfies:

po 1 $a \leq a$ for all $a \in \mathscr{M}$ (Reflexivity)
po 2 if $a \leq b$ and $b \leq a$ then $a = b$ (Antisymmetry).
po 3 if $a \leq b$ and $b \leq c$ then $a \leq c$ (Transitivity)

Note that there may be cases where not every two elements in M are relatable via $\leq$, i.e. $a \leq b$ or $b \leq a$, which is why we call the set "partially ordered". We say that a is comparable to b if $a \leq b$ or $b \leq a$, and incomparable if they are not comparable.

DEFINITION. A *totally ordered* set or a *chain* is a non-empty partially ordered set where every two elements of the set are comparable.

DEFINITION. An *upper bound* of a subset U of a partially ordered set M is an element $u \in M$ such that

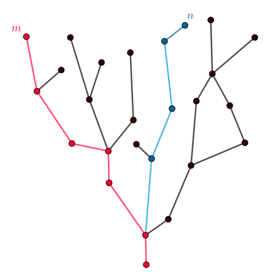
$$x \leq u, \qquad \forall x \in U$$

(such a bound may or may not exists). A *maximal element* of U is an element $m \in U$ such that

$$m \leq x \text{ and } x \in U \;\Rightarrow\; x = m$$

(as before, U may not have a maximal element). In particular, a maximal element of M is an element $m \in M$ such that

$$m \leq x \;\Rightarrow\; x = m.$$



A partially ordered set that is not a chain/totally ordered set. Nodes that are connected by an edge are comparable with the higher node being "larger". Two chains, one in red and one in light blue, are emphasised. Both have a maximal element - m and n respectively.

THEOREM (Zorn's Lemma). Let $M \neq \emptyset$ be a partially ordered set. If every chain C in M has an upper bound then M has a maximal element.

2. Distances

2.1. Metrics.

DEFINITION. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}_+$ is called a *metric* if it satisfies the following conditions:

m 1 $d(x, y) = 0$ if and only if $x = y$ (Positivity).
m 2 $d(x, y) = d(y, x)$ for all $x, y \in X$ (Symmetry).
m 3 $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$ (Triangle inequality).

The couple (X, d) is called a *metric space*.

REMARK. Given a metric space (X, d) and a subset $M \subset X$ we induce a metric on M from X by restricting the function $d : X \times X \to \mathbb{R}_+$ to M. (M, d) is automatically a metric space.

2.2. Norms.

DEFINITION. A set $\mathscr{X}$ is called a vector space over a field $\mathbb{F}$ if there exist two algebraic operations

$$+ : \mathscr{X} \times \mathscr{X} \to \mathscr{X}, \qquad \cdot : \mathbb{F} \times \mathscr{X} \to \mathscr{X}$$

such that

- $x + y = y + x$ for all $x, y \in \mathscr{X}$.
- $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathscr{X}$.
- $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ for all $\alpha, \beta \in \mathbb{F}$, $x \in \mathscr{X}$.
- $1 \cdot x = x$ for all $x \in \mathscr{X}$.
- $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ for all $\alpha \in \mathbb{F}$, $x, y \in \mathscr{X}$.
- $(\alpha + \beta)\, x = \alpha \cdot x + \beta \cdot x$ for all $\alpha, \beta \in \mathbb{F}$ and $x \in \mathscr{X}$.

Moreover, there exists an element $\mathbf{0} \in \mathscr{X}$ such that for any $x \in \mathscr{X}$

$$x + \mathbf{0} = x$$

and for all $x \in \mathscr{X}$ there exists an element $(-x) \in \mathscr{X}$ such that

$$x + (-x) = \mathbf{0}$$

When the difference between the elements of the field $\mathbb{F}$, known as *scalars*, and vectors is clear we usually write αx instead of $\alpha \cdot x$. We will also write 0 instead of $\mathbf{0}$ for the zero vector whenever no confusion between it and the additive zero of the underlying field arises.

DEFINITION. Given a vector field $\mathscr{X}$ over a field $\mathbb{F}$ and a subset $\mathscr{M} \subset \mathscr{X}$ we say that $\mathscr{M}$ is a *linear subspace*, or *subspace* in short, if $\mathscr{M}$ is a vector field over $\mathbb{F}$.

REMARK. To check if a subset $\mathscr{M}$ of a vector field $\mathscr{X}$ is a subspace one only needs to check that

- $0 \in \mathscr{M}$.
- $\mathscr{M}$ is closed under addition, i.e. if $x, y \in \mathscr{M}$ then $x + y \in \mathscr{M}$.
- $\mathscr{M}$ is closed under scalar multiplication, i.e. if $x \in \mathscr{M}$ then $\alpha x \in \mathscr{M}$ for any $\alpha \in \mathbb{F}$.

DEFINITION. Let $\mathscr{X}$ be a vector space over $\mathbb{F}$, be it $\mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\|$: $\mathscr{X} \to \mathbb{R}_+$ is called a *norm* if it satisfies the following conditions:

n 1 $\|x\| \geq 0$ for all $x \in \mathscr{X}$ and $\|x\| = 0$ if and only if $x = 0$ (Positivity).

n 2 $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{F}$ and all $x \in \mathscr{X}$ (Norms are homogeneous on $\mathbb{R}_+$).

n 3 $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathscr{X}$ (Triangle inequality).

The couple $(\mathscr{X}, \|\cdot\|)$ is called a *normed space*.

REMARK. Given a normed space $(\mathscr{X}, \|\cdot\|)$ and a subspace $\mathscr{M} \subset \mathscr{X}$ we induce a norm on $\mathscr{M}$ from $\mathscr{X}$ by restricting the function $\|\cdot\| : \mathscr{X} \to \mathbb{R}_+$ to $\mathscr{M}$. $(\mathscr{M}, \|\cdot\|)$ is automatically normed space.

THEOREM. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. Define the function $d : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_+$ by

$$d(x, y) = \|x - y\|.$$

Then d is a metric on $\mathscr{X}$. We call it the metric induced by the norm $\|\cdot\|$.
Unless stated otherwise, the metric structure in a normed space will always be the one induced from the norm.

You might not have seen the following theorem, but we add it here for completion (we won't really use it in our module):

THEOREM. Let $(\mathscr{X}, d)$ be a metric space where $\mathscr{X}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then the metric d is induced by a norm if and only if

(i) $d(x, y) = d(x + z, y + z)$ for any $x, y, z \in \mathscr{X}$.

(ii) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for any $x, y \in \mathscr{X}$ and scalar α.

In that case the norm which induces the metric is given by

$$\|x\| = d(x, 0).$$

2.3. Inner products.

DEFINITION. Let $\mathscr{X}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ or $\mathbb{C}$ (respectively) is called an *inner product* if it satisfies the following conditions:

p 1 $\langle x, x\rangle \geq 0$ for all $x \in \mathscr{X}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$ (Positivity).

p 2 $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for any $x, y, z \in \mathscr{X}$ (Addition of the first component).

p 3 $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ for any $x, y \in \mathscr{X}$ and any scalar α (Scalar multiplication of the first component).

p 4 $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (Symmetry/Hermitian property).

The couple $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ is called an *inner product* space and sometimes a *pre-Hilbert space*.

REMARK. Given an inner product space $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ and a subspace $\mathscr{M} \subset \mathscr{X}$ we induce an inner product on $\mathscr{M}$ from $\mathscr{X}$ by restricting the function $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_+$ to $\mathscr{M}$. $(\mathscr{M}, \langle\cdot,\cdot\rangle)$ is automatically an inner product space.

THEOREM. Let $(\mathscr{X}, \langle\cdot,\cdot\rangle)$ be an inner product space. Define the function $\|\cdot\| : \mathscr{X} \to \mathbb{R}_+$ by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Then $\|\cdot\|$ is a norm on $\mathscr{X}$. We call it the norm induced by the inner product $\langle\cdot,\cdot\rangle$. Unless stated otherwise, the metric structure in a normed space will always be the one induced from the norm which is induced from the inner product, i.e.

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

THEOREM. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space over $\mathbb{R}$ or $\mathbb{C}$. Then the norm $\|\cdot\|$ is induced by an inner product if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{2.1}$$

In that case the inner product which induces the norm is given by

$$\langle x, y\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4} \tag{2.2}$$

when $\mathscr{X}$ is over $\mathbb{R}$ and

$$\langle x, y\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4} + i\left(\frac{\|x + iy\|^2 - \|x - iy\|^2}{4}\right) \tag{2.3}$$

when $\mathscr{X}$ is over $\mathbb{C}$. Equation (2.1) is known as the *parallelogram identity* while equations (2.2) and (2.3) are known as the *polarisation identities.*

3. Metric spaces

3.1. Open and Closed sets.

DEFINITION. Let (X, d) be a metric space. Then:

- For any $x \in X$ and any $r > 0$ we define the *open ball of radius r centred at x* to be the set

  $$B_r(x) = \{y \in X \mid d(x, y) < r\},$$

  and the *closed ball of radius r centred at x* to be the set

  $$\overline{B}_r(x) = \{y \in X \mid d(x, y) \le r\}.$$

- A set $U \subset X$ is called an *open set* if for any $x \in U$ there exists $\varepsilon > 0$ (which can and usually does depend on x) such that

  $$B_\varepsilon(x) \subset U.$$

- A set $C \subset X$ is called a *closed set* if $C^c = X \setminus C$ is an open set.

THEOREM. Let (X, d) be a metric space. Then:

(i) For any $x \in X$ and $r > 0$ the open ball $B_r(x)$ is an open set.
(ii) A union, countable of uncountable, of open sets is an open set.
(iii) Finite intersections of open sets is an open set (this is not always true when we pass to countable or uncountable intersections).

REMARK. As a closed set C is a set such that C^c is open we have that

- An intersection, countable of uncountable, of closed sets is a closed set.
- Finite unions of closed sets is a closed set (this is not always true when we pass to countable or uncountable unions).

In Complex Analysis II you have defined the interior, closure and boundary of a set. It is sometimes very convenient to associate these notions to points. This is expressed in the next definition (parts of which you have seen), with the additional notion of accumulation points and the derived set:

DEFINITION. Let (X, d) be a metric space and let $U \subset X$ be a given set. Then

(i) We say that x is an *interior point* of U if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

(ii) We say that x is an *accumulation point* of U if every open ball centred at x contains a point $y \in U$ that is not x. In other words

$$(B_\varepsilon(x) \setminus \{x\}) \cap U \neq \emptyset \qquad \text{for all } \varepsilon > 0.$$

(iii) We say that x is a *boundary point* of U if every open ball that contains x contains a point from U and a point from U^c. In other words

$$B_\varepsilon(x) \cap U \neq \emptyset \quad \text{and} \quad B_\varepsilon(x) \cap U^c \neq \emptyset \qquad \text{for all } \varepsilon > 0.$$

(iv) Given a set U we define:
  - *The interior of* U, U° or $\text{int}(U)$, is the set of all interior points of U.
  - *The derived set of* U, U', is the set of all accumulation points of U.
  - *The boundary of* U, ∂U, is the set of all the boundary points of U.
  - *The closure of* U, $\overline{U}$ is the set $U \cup \partial U$.

REMARK. We notice that

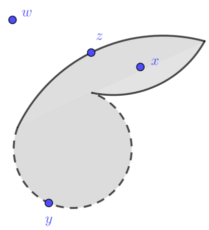
- Interior points are points in U that have a whole open ball around them that is also in U.
- Accumulation points are points that can be in or out of U but any *punctured* open ball around them must contain some point from U.
- Boundary points are points that "see" both U and U^c. Any open ball around such point contains a point from U, *which can be x itself,* and a point from U^c, *which, again, can be x itself.* An extreme case to consider is that of an *isolated point*: an isolated point x is such that there exists $\varepsilon_0 > 0$ with

$$B_{\varepsilon_0}(x) \cap U = \{x\}$$

  meaning that there is an open ball around x whose intersection with U is the point x.

We see that

- Any interior point is an accumulation point but not a boundary point.
- Any accumulation point that is not an interior point is a boundary point.
- Any isolated point is a boundary point, but is not an interior or accumulation point.
- A boundary point can never be an interior point.



Classification of points: x is an interior point, x, y and z are accumulation points, y, z and w are boundary points (w is isolated).

THEOREM. Let (X,d) be a metric space and let $U \subset X$ be given. Then

(i) U^{o} is an open set. Moreover, U is open if and only if $U = U^{o}$, or equivalently if every point in U is interior.

(ii) U^{o} is the largest open set contained in U, i.e. if $O \subset U$ and O is open then $O \subset U^{o}$.

(iii) $\overline{U}$ is a closed set. Moreover, U is closed if and only if $\overline{U} = U$.

(iv) $\overline{U} = U \cup U'$. As such, U is closed if and only if $U' \subset U$.

(v) $\overline{U}$ is the smallest closed set containing U, i.e. if $U \subset C$ and C is closed then $\overline{U} \subset C$.

3.2. Continuity of functions.

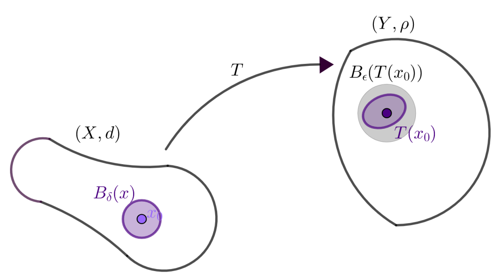
DEFINITION. Let (X,d) and (Y,ρ) be two metric spaces. We say that a function $T : X \to Y$ is *continuous at the point* $x_0 \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,x_0) < \delta$ then

$$\rho(T(x),T(x_0)) < \varepsilon.$$

We say that T is *continuous on* X if it is continuous in every point of X.

We have an alternative definition for continuity at a point:

THEOREM. Let (X,d) and (Y,ρ) be two metric spaces. A function $T : X \to Y$ is continuous at a point $x_0 \in X$ if and only for every $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\delta(x_0) \subset T^{-1}(B_\varepsilon(T(x_0)))$.



The image of the purple ball $B_\delta(x_0)$ on the left is the purple set on right, which is contained in the ball $B_\varepsilon(T(x_0))$.

More generally, one can show that:

THEOREM. Let (X,d) and $\left(Y,\rho\right)$ be two metric spaces.

(i) A function $T: X \to Y$ is continuous at a point x_0 if and only if for any open set $U \subset Y$ that contains $T(x_0)$ we have that $T^{-1}(U)$ contains a open set around x_0.

(ii) A function $T: X \to Y$ is continuous if and only if for any open set $U \subset Y$ we have that $T^{-1}(U)$ is open in X.

Continuity can also be defined on a subset of a metric space, usually the domain of the function:

DEFINITION. Let (X,d) and $\left(Y,\rho\right)$ be two metric spaces. We say that a function $T: \mathscr{D}(T) \subset X \to Y$ is *continuous at the point* $x_0 \in \mathscr{D}(T)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathscr{D}(T)$ and $d(x,x_0) < \delta$ then $\rho(T(x), T(x_0)) < \varepsilon$, or equivalently

$$B_\delta(x_0) \cap \mathscr{D}(T) \subset T^{-1}(B_\varepsilon(T(x_0))).$$

A criteria with open sets can also be given:

THEOREM. Let (X,d) and $\left(Y,\rho\right)$ be two metric spaces and let $T: M \subset \mathscr{D}(T) \subset X \to Y$ be given.

(i) T is continuous at a point $x_0 \in M$ if and only if for any open set $U \subset Y$ that contains $T(x_0)$ we have that $T^{-1}(U)$ contains a set of the form $M \cap V$ where $V \subset X$ is a open set that contains x_0.

(ii) T is continuous on M if and only if for any open set $U \subset Y$ we have that $T^{-1}(U)$ is of the form $M \cap V$ for some open set $V \subset X$.

3.3. Converging sequences.

DEFINITION. Let (X,d) be a metric space. We say that a sequence of elements in X, $\{x_n\}_{n\in\mathbb{N}}$, *converges* to the element $x \in X$ if the real sequence $\{d(x_n,x)\}_{n\in\mathbb{N}}$ converges to zero as n goes to infinity, i.e. if for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have that $d(x_n,x) < \varepsilon$.

REMARK. In normed spaces we have that: $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ converges to $x \in \mathscr{X}$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\|x_n - x\| < \varepsilon.$$

We have an alternative definition for convergence:

THEOREM. Let (X,d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements in X. Then $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if and only if for any open set $U \subset X$ we have that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have that $x_n \in U$.

Convergence can give an alternative definition to closedness of sets and continuity of functions:

THEOREM. Let (X,d) be a metric space and let $U \subset X$ be a given set. Then

(i) $x \in \overline{U}$ if and only if there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}} \subset U$ that converges to x.

(ii) U is closed if and only if every converging sequence of points from U converges to a point in U.

THEOREM. Let (X,d) and (Y,ρ) be two metric spaces and let $T : X \to Y$ be given. Then the following are equivalent:

(i) T is continuous at x_0.

(ii) For any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ that converges to x_0 we have that

$$\lim_{n\to\infty} T(x_n) = T(x_0)$$

The above can be extended to the case where the function is defined on a subset of the space:

THEOREM. Let (X,d) and (Y,ρ) be two metric spaces and let $T : M \subset \mathscr{D}(T) \subset X \to Y$ be given. Then the following are equivalent:

(i) T is continuous at $x_0 \in M$ (on M).

(ii) For any sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ that converges to $x_0 \in M$ we have that $\lim_{n\to\infty} T(x_n) = T(x_0)$.

THEOREM. Let (X,d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a given sequence. If $\{x_n\}_{n\in\mathbb{N}}$ converges to x then:

(i) $\{x_n\}_{n\in\mathbb{N}}$ is bounded, i.e. there exists $x_0 \in X$ and $M(x_0) > 0$ such that

$$\sup_{n\in\mathbb{N}} d(x_n, x_0) \le M(x_0).$$

In fact, the above is true for *any* $x_0 \in X$ (though $M(x_0)$ will depend on x_0).

(ii) The limit is unique, i.e. if $\{x_n\}_{n\in\mathbb{N}}$ also converges to $y \in X$ then $x = y$.

(iii) Any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, $\{x_{n_k}\}_{k\in\mathbb{N}}$, also converges to x.

REMARK. In normed spaces the boundedness of converging sequences can be expressed in the following way: We will say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ is bounded if there exists $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \|x_n\| \le M.$$

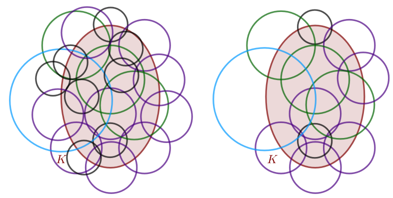
3.4. Compactness.

DEFINITION. Let (X,d) be a metric space. A set $K \subset X$ is called *sequentially compact* if for any sequence of elements in K, $\{x_n\}_{n\in\mathbb{N}}$, there exists a subsequence, $\{x_{n_k}\}_{k\in\mathbb{N}}$, that converges to an element x in K.

DEFINITION. Let (X,d) be a metric space. A set $K \subset X$ is called *compact* if for any collection of open sets $\{U_i\}_{i\in\mathscr{I}}$, with $\mathscr{I}$ being an index set of any cardinality, such that $K \subset \cup_{i\in\mathscr{I}} U_i$ we can find a finite collection of open sets $U_{i_1}, \dots, U_{i_n}$ such that

$$K \subset \cup_{i=1}^{n} U_i.$$

REMARK. The notion of compactness means that we are always able to remove redundancies from a large "open cover" of our set.



An "infinite" open cover of a set K on the left, and the finite open sub-cover of it on the right.

THEOREM. Let (X,d) be a metric space and let $K \subset X$. Then the following are equivalent:

(i) K is sequentially compact.
(ii) K is compact.

If any of the above holds we will use the notation "K is compact".

THEOREM. Let (X,d) be a metric space and let $K \subset X$ be compact. Then:

(i) K is a closed set.
(ii) K is a bounded set, i.e.

$$\sup_{x,y\in K} d\left(x,y\right) < \infty.$$

(iii) Any closed subset of a compact set is compact.
(iv) If $\left(Y,\rho\right)$ is another metric space and $T: X \to Y$ is a continuous function, then $T(K)$ is compact. Consequently, if $T: X \to \mathbb{R}$ then $T(K)$ is bounded, and T attains a maximum and minimum over K at some points in K.

REMARK. The converse of most of the statements above doesn't hold.

THEOREM (Heine-Borel). A set K in $\mathbb{R}^n$ is compact (with respect to the standard norm) if and only if it is closed and bounded.

REMARK. In any metric space (X,d) any compact set must be closed and bounded. The converse, however, is not true in general.

3.5. Cauchy sequences and completeness.

DEFINITION. Let (X,d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be given sequence in X. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* (or Cauchy in short) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$ we have that

$$d(x_n, x_m) < \varepsilon.$$

LEMMA. Let (X,d) be a metric space. Then any converging sequence, $\{x_n\}_{n\in\mathbb{N}}$, is Cauchy.

We will mention in class the following useful theorem:

THEOREM. Let (X,d) be a metric space and let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be Cauchy. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

Definition. We say that a metric space (X,d) is *complete* if every Cauchy sequence in X converges (to an element of X).

Theorem. Let (X,d) be a complete metric space and let M be a subset of X. Then (M,d) is complete if and only if M is closed.

4. Density and Separability

The notions of density and separability were introduced in Analysis III for normed spaces. They are, in fact, metric notions and we will define them as such here.

Definition. Let (X,d) be a metric space. We say that a set A is dense in X if for any $x \in X$ and any $\varepsilon > 0$, there exists $a_{x,\varepsilon} \in A$ such that

$$d\left(a_{x,\varepsilon}, x\right) < \varepsilon.$$

Remark. In normed spaces the above translates to: A set A is dense in a normed space $(\mathcal{X}, \|\cdot\|)$, if for any $x \in \mathcal{X}$ and any $\varepsilon > 0$, there exists $a_{x,\varepsilon} \in A$ such that

$$\left\| x - a_{x,\varepsilon} \right\| < \varepsilon.$$

Theorem. Let (X,d) be a metric space and let $A \subset B \subset X$ be given. If A is dense in B and B is dense in X then A is dense in X.

We will mention in class the following useful theorem:

Theorem. Let (X,d) be a metric space. Then A is dense in X if and only if $\overline{A} = X$.

Definition. Let (X,d) be a metric space. We say that X is separable if there exists a *countable* set $A \subset X$ that is dense in X.

5. Banach spaces

Definition. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. We say that $\mathcal{X}$ is a *Banach space* if it is complete under the metric induced by $\|\cdot\|$.

6. Linear functionals

Definition. Let $\mathcal{X}$ be a vector space over a field $\mathbb{F}$. A map $f : \mathcal{X} \to \mathbb{F}$ is called a *linear functional* if

$$f\left(\alpha x + \beta y\right) = \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{F}$.

Definition. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space over a field $\mathbb{F}$, be it $\mathbb{R}$ or $\mathbb{C}$, and let $f : \mathcal{X} \to \mathbb{F}$ be a linear functional. We say that f is a *bounded* linear functional if thee exists $M > 0$ such that

$$\left| f(x) \right| \leq M \|x\|$$

for any $x \in \mathcal{X}$. For any bounded linear functional f we define

$$\left\| f \right\| = \inf\left\{ M > 0 \mid \left| f(x) \right| \leq M \|x\|, \ \forall x \in \mathcal{X} \right\}.$$

In Analysis III you have defined the above over $\mathbb{R}$ only, but the same definition holds over $\mathbb{C}$. We will expand on this in our class.

7. Hilbert spaces

7.1. Basic properties.

DEFINITION. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. We say that $\mathcal{H}$ is a *Hilbert space* if it is complete under the metric induced by $\langle\cdot,\cdot\rangle$.

THEOREM (Cauchy-Schwartz inequality). Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Then for any $x, y \in \mathcal{H}$ we have that

$$\left|\left\langle x, y\right\rangle\right| \leq \|x\| \left\|y\right\|.$$

THEOREM. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. Then $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ is a continuous function, i.e. if $\{x_n\}_{n\in\mathbb{N}}$ and $\left\{y_n\right\}_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}$ that converge to $x \in \mathcal{H}$ and $y \in \mathcal{H}$ respectively then

$$\left\langle x_n, y_n\right\rangle \underset{n\to\infty}{\longrightarrow} \left\langle x, y\right\rangle.$$

7.2. Orthogonality.

DEFINITION. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space.

(i) We say that x is *orthogonal* to y, and write $x \perp y$ if

$$\left\langle x, y\right\rangle = 0.$$

(ii) We say that a set M is orthogonal if every two elements of it are orthogonal.

(iii) We say that two sets, A and B, are orthogonal if for any $x \in A$ and $y \in B$ we have that $x \perp y$.

(iv) Given a subset M of $\mathcal{H}$ we define the *orthogonal complement* of M, $M^\perp$ to be the set

$$M^\perp = \left\{x \in \mathcal{H} \;\middle|\; x \perp y,\ \forall y \in M\right\}$$

THEOREM (Pythagoras' theorem). Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. If $x_1, \dots, x_n \in \mathcal{H}$ and $x_i \perp x_j$ for $i \neq j$ then

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

THEOREM. Let $\mathcal{H}$ be an inner product space and let M be a subset of $\mathcal{H}$. Then $M^\perp$ is a closed subspace of $\mathcal{H}$.

THEOREM. Let $\mathcal{H}$ be an inner product space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathcal{H}$ (i.e. it is an orthonormal set). Then

$$\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \leq \|x\|^2.$$

This inequality is knows as *Bessel's inequality.*

THEOREM. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then:

(i) For any $x \in \mathscr{H}$ there exists a unique vector $x_{\parallel}$ in $\mathscr{M}$ such that

$$\left\| x - x_{\parallel} \right\| = \inf_{v \in \mathscr{M}} \|x - v\| = \min_{v \in \mathscr{M}} \|x - v\|.$$

We denote this vector by $P_{\mathscr{M}}(x)$ and call it the orthogonal projection of x on $\mathscr{M}$.

(ii) For any $x \in \mathscr{H}$ we have that $x - P_{\mathscr{M}}(x) \in \mathscr{M}^{\perp}$.

(iii) $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$.

REMARK. We say that a vector space V is the *direct sum* of the subspaces U and W, and write

$$V = U \oplus W,$$

if

- $V = U + W = \{u + w \mid u \in U,\ w \in W\}$.
- $U \cap W = \{0\}$.

7.3. Orthonormal basis.

DEFINITION. Let $\mathscr{H}$ be a Hilbert space. We say that a set $\mathscr{B} = \{e_\alpha\}_{\alpha \in \mathscr{I}}$ is an orthonormal basis for $\mathscr{H}$ if $\mathscr{B}$ is orthonormal and every $x \in \mathscr{H}$ satisfies

$$x = \sum_{\alpha \in \mathscr{I}} \langle x, e_\alpha \rangle e_\alpha.$$

We will discuss the above in more details (in particular, what does it mean that x is $\sum_{\alpha \in \mathscr{I}} \langle x, e_\alpha \rangle e_\alpha$) in more details in our module. We will show that there is a relatively straight forward definition if $\mathscr{I} = \mathbb{N}$ and when it is uncountable we will use the following theorem, which was shown in Analysis III:

THEOREM. Let $\mathscr{H}$ be an inner product space and let $\mathscr{B} = \{e_\alpha\}_{\alpha \in \mathscr{I}}$ be orthonormal. If $\mathscr{I}$ is uncountable, then for any $x \in \mathscr{H}$ we have that $\langle x, e_\alpha \rangle \neq 0$ for at most a countable subset of $\mathscr{B}$, $\left\{e_{\alpha_n}\right\}_{n \in \mathbb{N}}$.

THEOREM. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{B} = \{e_\alpha\}_{\alpha \in \mathscr{I}}$ be an orthonormal set in $\mathscr{H}$. Then the following are equivalent:

(i) $\mathscr{B}$ is an orthonormal basis, i.e. for any $x \in \mathscr{H}$ we have that

$$x = \sum_{\alpha \in \mathscr{I}} \langle x, e_\alpha \rangle e_\alpha.$$

(ii) For any $x \in \mathscr{H}$ we have that

$$\|x\|^2 = \sum_{\alpha \in \mathscr{I}} |\langle x, e_\alpha \rangle|^2.$$

The above is known as *Parseval's identity.*

(iii) $\mathscr{B}^{\perp} = \{0\}$.

THEOREM. Every non-trivial Hilbert space has an orthonormal basis. Moreover, if $\mathscr{B}'$ is an orthonormal set in $\mathscr{H}$ then there exists an orthonormal basis $\mathscr{B}$ for $\mathscr{H}$ such that $\mathscr{B}' \subset \mathscr{B}$.

7.4. Riesz's representation theorem.

THEOREM (Riesz's representation theorem for Hilbert spaces). Let $\mathscr{H}$ be a Hilbert space and let f be a bounded linear functional on $\mathscr{H}$. Then there exists a unique $y \in \mathscr{H}$ such that

$$f(x) = \langle x, y \rangle .$$

Moreover, $\|f\| = \|y\|$.

8. Lebesgue theory

8.1. Lebesgue measurable sets and the Lebesgue measure. You have only considered Lebesgue measurable sets on $\mathbb{R}$, but one can easily extend it to $\mathbb{R}^n$.

DEFINITION. An $n-$dimensional rectangle $R \subset \mathbb{R}^n$ is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

with $\{a_i\}_{i=1,\ldots,n}, \{a_b\}_{i=1,\ldots,n} \subset \mathbb{R}$ and where $-\infty < a_i \leq b_i < \infty$ for all $i = 1, \ldots, n$. The *volume* of R is defined as

$$\mathrm{vol}(R) = (b_1 - a_1)(b_2 - a_2) \ldots (b_1 - a_1) .$$

We denote the set of $n-$dimensional rectangles by $\mathcal{R}(\mathbb{R}^n)$.

DEFINITION. Given a set $E \subset \mathbb{R}^n$, the *Lebesgue outer measure* of E is defined as

$$|E|^* = \inf \left\{ \sum_{i \in \mathbb{N}} \mathrm{vol}(R_i) \mid E \subset \cup_{i \in \mathbb{N}} R_i,\ R_i \in \mathcal{R}\left(\mathbb{R}^n\right) \text{ for all } i \in \mathbb{N} \right\}$$

DEFINITION. A set $E \subset \mathbb{R}^n$ is Lebesgue measurable if for any set $A \subset \mathbb{R}^n$ we have that

$$|E|^* = |E \cap A|^* + \left|E \cap A^c\right|^* .$$

In that case we define the Lebesgue measure of E as $|E| = |E|^*$. We denote the set of measurable sets on $\mathbb{R}^n$ as $\mathfrak{L}(\mathbb{R}^n)$.

THEOREM. $\mathfrak{L}(\mathbb{R}^n)$ is a $\sigma-$algebra, i.e.

- $\mathbb{R}^n \in \mathfrak{L}(\mathbb{R}^n)$.
- If $E \in \mathfrak{L}(\mathbb{R}^n)$ then $E^c \in \mathfrak{L}(\mathbb{R}^n)$.
- If $\{E_i\}_{i \in \mathbb{N}} \subset \mathfrak{L}(\mathbb{R}^n)$ then $\cup_{i \in \mathbb{N}} E_i \in \mathfrak{L}(\mathbb{R}^n)$.

The Lebesgue measure is a non-negative measure on $\mathfrak{L}(\mathbb{R}^n)$, i.e.

- $|E| \geq 0$ for any $E \in \mathfrak{L}(\mathbb{R}^n)$.
- $|\varnothing| = 0$.
- If $\{E_i\}_{i \in \mathbb{N}} \subset \mathfrak{L}(\mathbb{R}^n)$ are pairwise disjoint (i.e. $E_j \cap E_i = 0$ when $i \neq j$) then

$$|\cup_{i \in \mathbb{N}} E_i| = \sum_{i \in \mathbb{N}} |E_i| .$$

The Lebesgue measure enjoys additional regularity properties which we will not mention here.

8.2. Lebesgue measurable functions.

DEFINITION. A function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is called *Lebesgue measurable on a measurable set E* if for any $\alpha \in \mathbb{R}$ the set $f^{-1}(-\infty, \alpha) \cap E$ is Lebesgue measurable.

REMARK. There are many equivalent definition which includes various other sets.

THEOREM. We have the following:

- If f and g are measurable than so are$\alpha f + \beta g$ is measurable for any $\alpha, \beta \in \mathbb{R}$, and fg.
- If $f_1, \ldots, f_n$ are measurable then so are $\max\{f_1, \ldots, f_n\}$ and $\min\{f_1, \ldots, f_n\}$.
- If f is measurable then so is $|f|$.
- If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions then so are $\sup_{n\in\mathbb{N}}\{f_n\}$ and $\inf_{n\in\mathbb{N}}\{f_n\}$.
- If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions then so are $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$. Consequently, if $\{f_n\}_{n\in\mathbb{N}}$ converges poitnwise to f then f is measurable.

8.3. The Lebesgue integral.

DEFINITION. A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is called *simple* if there exists $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $E_1, \ldots, E_n \in \mathfrak{L}(\mathbb{R}^n)$ such that

$$\psi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

where χ_A is the *characteristic function* of the set A, i.e.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

The *Lebesgue integral* of a simple function ψ over a measurable set E is defined as

$$\int_E \psi(x)\,dx = \int_E \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)\,dx = \sum_{i=1}^{n} \alpha_i \,|E \cap E_i|\,.$$

DEFINITION. Given a non-negative measurable function f and a measurable set E we define

$$\int_E f(x)\,dx = \sup\left\{\int_E \psi(x)\,dx \mid \psi(x) \text{ is a simple function and } \psi \le f \text{ on } E\right\}.$$

For a general measurable function f we define

$$f_+ = \max\{f, 0\}, \qquad _- = \max\{-f, 0\}$$

and find that $f = f_+ - f_-$. When

$$\int_E f_+(x)\,dx, \int_E f_-(x)\,dx < \infty$$

we define the Lebesgue integral of f as

$$\int_E f(x)\,dx = \int_E f_+(x)\,dx - \int_E f_-(x)\,dx.$$

THEOREM. We have the following properties of the Lebesgue integral:

- For any measurable functions f, g and any $\alpha, \beta \in R$ we have that
$$\int_E \left(\alpha f(x) + \beta g(x)\right) dx = \alpha \int_E f(x) dx + \beta \int_E g(x) dx.$$
- if $f \leq g$ and f and g are measurable then $\int_E f(x) dx \leq \int_E g(x) dx$.
- If f is measurable then $\left|\int_E f(x) dx\right| \leq \int_E \left|f(x)\right| dx$.
- If E and F are disjoint measurable sets and f is measurable then
$$\int_{E\cup F} f(x) dx = \int_E f(x) dx + \int_F f(x) dx.$$

Everything above can be extended to functions $f : E \to \mathbb{C}$ by considering the decomposition $f = \mathrm{Re} f + i \mathrm{Im} f$.

THEOREM (Fatou's Lemma). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions over a measurable set E. Then

$$\int_E \liminf_{n\to\infty} f_n(x) dx \leq \liminf_{n\to\infty} \int_E f_n(x) dx.$$

In particular, if $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f then

$$\int_E f(x) dx \leq \liminf_{n\to\infty} \int_E f_n(x) dx.$$

THEOREM (Monotone convergence theorem). Let $\{f_n\}_{n\in\mathbb{N}}$ be a non-decreasing sequence of non-negative measurable functions over a measurable set E. Then, if $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f then

$$\lim_{n\to\infty} \int_E f_n(x) dx = \int_E f(x) dx.$$

THEOREM (Dominated convergence theorem). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions over a measurable set E. Assume that there exists a non-negative measurable function over E, g, such that $\left|f_n\right| \leq g$ for all $n \in \mathbb{N}$. If $\int_E g(x) dx < \infty$ and if $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f then

$$\lim_{n\to\infty} \int_E f_n(x) dx = \int_E f(x) dx.$$

9. L^p spaces

Basic properties.

DEFINITION. Given a measurable set $E \subset \mathbb{R}^n$ we define the space $L^p(E)$, where $1 \leq p < \infty$ as

$$L^p(E) = \left\{ f : E \to \mathbb{C} \mid f \text{ is measurable and} \int_E \left|f(x)\right|^p dx < \infty \right\} / \simeq .$$

where $\simeq$ is the equivalence relation $f \simeq g$ if there exists a set of measure zero N such that $f|_{N^c} = g|_{N^c}$. We say that f and g are equal *almost everywhere*.

We define the function $\|\cdot\|_p : L^p(E) \to \mathbb{R}_+$ by

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Definition. Given a measurable set $E \subset \mathbb{R}^n$ we say that a measurable function f is essentially bounded if there exists $M > 0$ and a measurable set N such that

$$|f(x)| \le M, \qquad \text{for all } x \in E \cap N^c.$$

We define

$$L^\infty(E) = \{ f : E \to \mathbb{C} \mid f \text{ is essentially bounded} \} / \simeq,$$

where $\simeq$ is the equivalence relation of equality almost everywhere.
We define the function $\|\cdot\|_\infty : L^\infty(E) \to \mathbb{R}_+$ by

$$\|f\|_{L^\infty(E)} = \operatorname*{ess\,sup}_{x \in E} |f(x)| = \inf \{ M > 0 \mid |\{ x \in E \mid |f(x)| > M \}| = 0 \}.$$

Definition. We say that $p, q \in [1, \infty]$ are *Hölder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1$$

where $\frac{1}{\infty}$ is to be understood as 0.

Theorem (Young's inequality). Let $p, q \in (1, \infty)$ be Hölder conjugate. Then for any $x, y \in \mathbb{C}$

$$|xy| \le \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

Theorem (Hölder inequality). Let $p, q \in [1, \infty]$ be Hölder conjugate. Then for any $f \in L^p(E)$and $g \in L^q(E)$ we have that

$$\int_E |f(x) g(x)| \, dx \le \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

Theorem (Minkowski's inequality). Given a measurable set $E \subset \mathbb{R}^n$ and $p \in [1, \infty]$ we have that for any $f, g \in L^p(E)$

$$\|f + g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)}.$$

Theorem. Given a measurable set $E \subset \mathbb{R}^n$ and $p \in [1, \infty]$ we have that $L^p(E)$ is a Banach space. For $p = 2$ it is in fact a Hilbert space with

$$\langle f, g \rangle_{L^2(E)} = \int_E f(x) \overline{g(x)} \, dx.$$

Theorem. Let $E \subset \mathbb{R}^n$ be a measurable set. Then $L^p(E)$ is separable for any $p \in [1, \infty)$.

Approximation by continuous functions.

THEOREM. Let $E \subset \mathbb{R}^n$ be a measurable set and let $p \in [1,\infty)$. For any $f \in L^p(E)$ and any $\varepsilon > 0$ we can find a continuous function on E, $g_{f,\varepsilon}$, such that

$$\left\| f - g_{f,\varepsilon} \right\|_{L^p(E)} < \varepsilon.$$

In other words, the space of $C(E) \cap L^p(E)$, where $C(E)$ stands for the continuous functions on E, is dense in $L^p(E)$.

REMARK. In general, $L^\infty(E)$ is *not* separable.

9.1. Reisz's representation theorem for L^p.

THEOREM (Riesz's representation theorem for L^p). Let $E \subset \mathbb{R}^n$ be a measurable set and let $p \in [1,\infty)$ be given. For any bounded linear functional $\mathfrak{I} : L^p(E) \to \mathbb{C}$ there exists $g \in L^q(E)$, where q is the Hölder conjugate of p, such that

$$\mathfrak{I}(f) = \int_E f(x)\overline{g(x)}dx.$$

Moreover, $\|\mathfrak{I}\| = \|g\|_{L^q(E)}$.