#Functional

Exercise 1. Consider the space $(C[0,1], \|\cdot\|_\infty)$ and the sets

$$C^1[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuously differentiable on } [0,1]\},$$

$$C_0^1[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuously differentiable on } [0,1],\ f(0) = f(1) = 0\}.$$

(i) Show that $C^1[0,1]$ and $C_0^1[0,1]$ are subspaces of $C[0,1]$.

(ii) Show that $C_0^1[0,1]$ is a subspace of $C^1[0,1]$.

Define a function $\|\cdot\|_{C^1} : C^1[0,1] \to \mathbb{R}_+$ by

$$\|f\|_{C^1} = \|f'\|_\infty.$$

(iii) Show that $\|\cdot\|_{C^1}$ is not a norm on $C^1[0,1]$ but is a norm on $C_0^1[0,1]$.

(iv) Show that $(C_0^1[0,1], \|\cdot\|_{C^1})$ is a Banach space.

Sol: The zero function belongs to both $C^1[0,1]$ and $C_0^1[0,1]$.

From Analysis I we know that if $f, g \in C^1[0,1]$ then $f+g \in C^1[0,1]$.

If $f, g \in C_0^1[0,1]$

$(f+g)(0) = f(0) + g(0) = 0$

$(f+g)(1) = 0$

$\Rightarrow f+g \in C_0^1[0,1]$.

Similarly, if $f \in C^1[0,1]$ and $\alpha \in \mathbb{F}$ then (Analysis I) $\alpha f \in C^1[0,1]$.

If $f \in C_0^1[0,1]$ then

$(\alpha f)(0) = \alpha f(0) = 0$

$(\alpha f)(1) = 0$

$\Longrightarrow \alpha f \in C_0^1[0,1]$.

Both subsets are subspaces.

Moreover since $C_0^1[0,1] \subset C^1[0,1]$ and $C_0^1[0,1]$ is a vector space we have that $C_0^1[0,1]$ is a subspace of $C^1[0,1]$.

(ii) $\|f\|_{C^1} = 0 \Longleftrightarrow \|f'\|_\infty = 0$

$\underset{\substack{\|\cdot\|_\infty \text{ is} \\ \text{a norm} \\ \text{on } C[0,1]}}{\Longleftrightarrow} |f'| = 0 \Longleftrightarrow f' = 0$

$\underset{\substack{\text{from} \\ \text{Analysis}}}{\Longleftrightarrow} f = \text{const}$

$\|f\|_{C^1} = 0$ doesn't imply, in general, that $f = 0$. Indeed $f \equiv 1$ satisfies $\|f\|_{C^1} = 0$

If $f \in C_0^1[0,1]$ and $f \equiv c$

then $c = f(0) = 0$

so on $C_0^1[0,1]$ $\|f\|_{C^1} = 0 \Rightarrow f \equiv 0.$

$\|\cdot\|_{C^1}$ is not a norm on $C^1[0,1]$.

on $C_0^1[0,1]$ we have $\forall \alpha \in \mathbb{F}$

$$\|\alpha f\|_{C^1} = \|(\alpha f)'\|_\infty = \|\alpha f'\|_\infty$$

$$\underset{\substack{\|\cdot\|_\infty \text{ is} \\ \text{a norm}}}{=} |\alpha| \, \|f'\|_\infty = |\alpha| \, \|f\|_{C^1}$$

similarly $\forall f, g \in C_0^1[0,1]$

$$\|f+g\|_{C^1} = \|(f+g)'\|_\infty = \|f'+g'\|_\infty$$

$$\underset{\substack{\|\cdot\|_\infty \text{ is} \\ \text{a norm}}}{\leq} \|f'\|_\infty + \|g'\|_\infty = \|f\|_{C^1} + \|g\|_{C^1}$$

$\Rightarrow \|\cdot\|_{C^1}$ is a norm on $C_0^1[0,1]$.

(iv) Let $\{f_n\}_{n \in \mathbb{N}} \subset C_0^1[0,1]$ be Cauchy in $\|\cdot\|_{C^1}$. i.e $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ s.t

$\forall n, m > n_0$

$$\underbrace{\|f_n - f_m\|_{C^1}}_{\|f_n' - f_m'\|_\infty} < \varepsilon$$

We find that $\{f_n'\}_{n\in\mathbb{N}} \subset C[0,1]$ is Cauchy in $\|\cdot\|_\infty$. $(C[0,1], \|\cdot\|_\infty)$ is Banach $\Rightarrow$ $\exists g \in C[0,1]$ s.t

$$f_n' \xrightarrow{\|\cdot\|_\infty} g.$$

g is the candidate for the derivative of the limit of $\{f_n\}$ in $\|\cdot\|_{C^1}$. Define

$$f(x) = \int_0^x g(t)\,dt$$

From fundamental theorem of calculus $f \in C^1[0,1]$. $f(0) = \int_0^0 \_\_ = 0$

if $f(1) = 0$ then $f \in C_0^1[0,1]$

and

$$\|f_n - f\|_{C^1} = \|f_n' - g\| \xrightarrow[n\to\infty]{} 0$$

Indeed

$$f(1) = \int_0^1 g(t)\,dt = \int_0^1 \lim_{n\to\infty} f_n'(t)\,dt$$

$$\underset{\text{uniform conv}}{=} \lim_{n\to\infty} \int_0^1 f_n'(t)\,dt = \lim_{n\to\infty} (\underbrace{f_n(1)}_{0} - \underbrace{f_n(0)}_{0}) = 0.$$

Exercise 2. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space.

(i) Show that if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{X}$ then one can extract a subsequence of it, $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\left\| x_{n_k} - x_{n_{k+1}} \right\| < \frac{1}{2^k}.$$

We say that a series $\sum_{n\in\mathbb{N}} x_n$ *converges* in $\mathscr{X}$ is the sequence of partial sums, $\{S_N\}_{N\in\mathbb{N}}$, defined as

$$S_N = \sum_{n=1}^{N} x_n$$

converges in $\mathscr{X}$. We say that a series $\sum_{n\in\mathbb{N}} x_n$ *converges absolutely* in $\mathscr{X}$ if $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.

(ii) Show that if $(\mathscr{X}, \|\cdot\|)$ is a Banach space then every absolutely converging series converges.

(iii) Show that if $(\mathscr{X}, \|\cdot\|)$ is a normed space where every absolutely converging series converges, then $(\mathscr{X}, \|\cdot\|)$ is a Banach space.
Hint: You may use the fact that any Cauchy sequence of a converging subsequence converges to the same limit as the original sequence.

See typed solution.

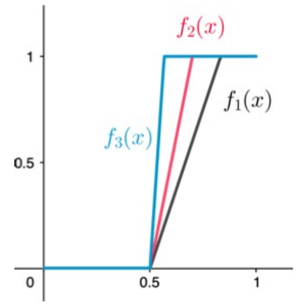
Exercise 3 *(If time permits)*. Consider the vector space $C[0,1]$ and define the function $\|\cdot\|_1 : C[0,1] \to \mathbb{R}_+$ by

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

You may assume without proof that $(C[0,1], \|\cdot\|_1)$ is a normed space. Show that it is not a Banach space.

Hint: Consider the sequence

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ n\left(x - \frac{1}{2}\right) & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$



and use the fact that for any $0 \le a \le b \le 1$ *we have that for any* $g \in C[0,1]$

$$\int_a^b |g(x)|\, dx \le \int_0^1 |g(x)|\, dx.$$

Sol: Let $n, m \in \mathbb{N}$. w.l.o.g $m > n$

$$\int_0^1 |f_n(x) - f_m(x)|\, dx$$

$$= \int_{1/2}^{1/2 + 1/m} |f_n(x) - f_m(x)|\, dx \le \int_{1/2}^{1/2 + 1/n} 2\, dx$$

$0 \le f_j(x) \le 1$

$$= \frac{2}{n}$$

For any $\varepsilon > 0$ if $n, m > 2/\varepsilon$

$$\| f_n - f_m \|_1 < \varepsilon$$

Assume that $f_n \xrightarrow{\|\cdot\|_1} f \in C[0,1]$.

$$0 \leq \int_0^{1/2} |f(x)| dx \leq \int_0^1 |f_n(x) - f(x)| dx \xrightarrow[n \to \infty]{} 0$$

$f_n|_{[0,1/2]} = 0$

$\forall n$

$$\Rightarrow \int_0^{1/2} |f(x)| dx = 0$$

since f is cont. $f \equiv 0$ on $[0, 1/2]$.

For any $\delta > 0$ $\exists n_0$ s.t if $n \geq n_0$

$$1/2 + 1/n < 1/2 + \delta$$

$\forall n \geq n_0$,

$$0 \leq \int_{1/2+\delta}^{1} |1 - f(x)| dx \leq \int_0^1 |f_n(x) - f(x)| dx \xrightarrow[n \to \infty]{} 0$$

$f_n|_{[1/2+\delta, 1]} = 1$

for $n \geq n_0$

$\Rightarrow$ since f is cont

$$f \equiv 1 \quad \text{on} \quad [1/2+\delta, 1]$$

for any $\delta > 0$

$$\Rightarrow \quad f \equiv 1 \quad \text{on} \quad (1/2, 1]$$

this implies that

$$0 = f(1/2) = \lim_{x \to 1/2^+} f(x) = 1$$

Contradiction.