#Functional

Exercise 1. Show that $\mathscr{B} = \{1, x, \ldots\}$ is not a Schauder basis for $C[a,b]$.

with $\|\cdot\|_\infty$.

Solution: If $\mathscr{B}$ is Schauder then for any $f \in C[a,b]$ we can find a sequence of scalars $\{a_n(f)\}_{n\in\mathbb{N}}$ s.t

$$S_N = \sum_{n=0}^{N} a_n(f)\, x^n$$

converges to f uniformly on $[a,b]$.

This implies that

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\, x^n$$

i.e. f can be represented by a power series on $[a,b]$. This implies that f is analytic in (a,b)

But not every $f \in C[a,b]$ is diff. on (a,b), let alone analytic. Contradiction.

Exercise 2. Show that if $\mathscr{H}$ is an inner product space that has an uncountable orthonormal set then it can't be separable.

Solution: Let $B = \{e_\alpha\}_{\alpha \in I}$ be the uncountable orthonormal set in H.

$\forall \alpha \neq \beta$

$$\|e_\alpha - e_\beta\|^2 \underset{\text{Pythagoras}}{=} \|e_\alpha\|^2 + \|e_\beta\|^2 = 2$$

$\rightarrow$

$$\|e_\alpha - e_\beta\| = \sqrt{2}$$ — uniform distance between any two elements of B

since B is uncountable, a theorem from class guarantees that H is not separable.

Exercise 3. Consider the Schauder basis $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ for $\ell_p(\mathbb{N})$, $p \in [1,\infty)$, defined by

$$(\boldsymbol{e}_n)_k = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}.$$

Define the vectors

$$\boldsymbol{x}_n = \begin{cases} \boldsymbol{e}_1 & n = 1 \\ \boldsymbol{e}_n - \boldsymbol{e}_{n-1} & n \geq 2 \end{cases}.$$

Show that $\widetilde{\mathscr{B}} = \{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ is a Schauder basis for $\ell_1(\mathbb{N})$.

Heuristic:

$$x_1 = e_1$$
$$x_2 = e_2 - e_1$$
$$x_3 = e_3 - e_2$$
$$\vdots$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ -1 & 1 & 0 & 0 & \cdots & \\ 0 & -1 & 1 & 0 & \cdots & \cdots \\ & & & \ddots & & \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \end{pmatrix}$$

Solution: $\tilde{B} = \{x_n\}_{n\in\mathbb{N}}$. We claim that $\tilde{B}$ is independent. It is sufficient to show that $x_1, \dots, x_k$ are independent for any $k \in \mathbb{N}$.

Assume $\sum_{i=1}^{k} \alpha_i x_i = 0$

$$\Rightarrow \quad \alpha_1 x_1 + \sum_{i=2}^{k} \alpha_i x_i = 0$$

$$\Leftrightarrow$$

$$\alpha_1 e_1 + \underbrace{\sum_{i=2}^{k} \alpha_i (e_i - e_{i-1})}_{\sum_{i=2}^{k} \alpha_i e_i - \sum_{j=1}^{k-1} \alpha_{j+1} e_j} = 0$$

$$\longrightarrow \quad \sum_{i=1}^{k} \alpha_i e_i - \sum_{i=1}^{k-1} \alpha_{i+1} e_i = 0$$

$$\longleftrightarrow \quad \sum_{i=1}^{k-1} (\alpha_i - \alpha_{i+1}) e_i + \alpha_k e_k = 0$$

$\{e_1, \dots, e_k\}$ are independent $\longrightarrow$

$$\alpha_k = 0, \quad \alpha_i = \alpha_{i+1} \quad \forall\, i = 1, \dots, k-1$$

$\Rightarrow \quad \alpha_1 = \dots = \alpha_k = 0$ what we wanted.

We would like to find a partial sum approximation for any $\vec{a} \in \ell_1(\mathbb{N})$ using $\widetilde{B}$. Lets look at

$$S_N = \sum_{n=1}^{N} \alpha_n x_n \xrightarrow[N \to \infty]{?} \vec{a}$$

$$x_1 = e_1 \qquad x_2 = e_2 - e_1$$

$$e_2 = x_2 + e_1 = x_2 + x_1$$

$$x_3 = e_3 - e_2 = e_3 - x_2 - x_1$$

$$e_3 = x_1 + x_2 + x_3$$

$$\vdots$$

$$e_n = \sum_{i=1}^{n} x_i$$

We know that

$$S_N(\vec{a}) = \sum_{n=1}^{N} a_n e_n \xrightarrow{N\to\infty} \vec{a}$$

$$\left(\vec{a} = (a_1, a_2, \dots) \right)$$

$$S_N(\vec{a}) = \sum_{n=1}^{N} a_n \left(\sum_{i=1}^{n} x_i \right)$$

$$= \sum_{i=1}^{N} \left(\sum_{n=i}^{N} a_n \right) x_i$$

For large N $\sum_{n=i}^{N} a_n$ behaves like $\sum_{n=i}^{\infty} a_n$ which converges as $\vec{a} \in \ell_1(\mathbb{N})$

We consider

$$S_N = \sum_{n=1}^{N} \left(\sum_{j=n}^{\infty} a_j \right) x_n$$

<u>Claim</u> $S_N \xrightarrow[N\to\infty]{} \vec{a}$

$$S_N = \sum_{n=1}^{N} \left(\sum_{j=n}^{N} a_j + \sum_{j=N+1}^{\infty} a_j \right) x_n$$

$$= \underbrace{\sum_{n=1}^{N}\left(\sum_{j=n}^{N} a_j\right) x_n}_{S_N(\vec{a})} + \left(\sum_{j=N+1}^{\infty} a_j\right)\underbrace{\left(\sum_{n=1}^{N} x_n\right)}_{e_N}$$

$$\| S_N - S_N(\vec{a})\|_1 = \left|\sum_{j=N+1}^{\infty} a_j\right| \underbrace{\|e_N\|_1}_{1}$$

$$= \left|\sum_{j=N+1}^{\infty} a_j\right| \xrightarrow[N\to\infty]{} 0$$

since $\vec{a} \in \ell_1(\mathbb{N})$

Since $S_N(\vec{a}) \longrightarrow \vec{a}$ the above implies

that $S_N \longrightarrow \vec{a}$.

To conclude that $\mathcal{B}$ is Schauder

we need to prove that if

$$\hat{S}_N = \sum_{j=1}^{N} \alpha_j x_j$$

converges to $\vec{a}$ then $\alpha_j = \sum_{n=j}^{\infty} a_n$

$$\hat{S}_N = \sum_{j=1}^{N} \alpha_j x_j = \sum_{j=1}^{N-1} (\alpha_j - \alpha_{j+1}) e_j + \alpha_N e_N$$

Recall that for any $k \in \mathbb{N}$

$$|a_k - b_k| \leq \|\vec{a} - \vec{b}\|_1$$

$\Rightarrow \quad \forall \quad j = 1, \ldots, N-1$

$$|a_j - (\alpha_j - \alpha_{j+1})| \leq \|\vec{a} - \hat{S}_N\|_1$$

and
$$|a_N - \alpha_N| \leq \|\tilde{a} - \hat{S}_N\|_1$$
We assume $\|\tilde{a} - \hat{S}_N\|_1 \xrightarrow[N\to\infty]{} 0$

$\Rightarrow$ $\forall j \in \mathbb{N}$ fixed
$$0 \leq |a_j - (\alpha_j - \alpha_{j+1})| \leq \|a - \hat{S}_N\|_1 \xrightarrow[N\to\infty]{} 0$$
i.e $a_j = \alpha_j - \alpha_{j+1} \quad \forall j \in \mathbb{N}$

so
$$a_j = \alpha_j - \alpha_{j+1}$$
$$a_{j+1} = \alpha_{j+1} - \alpha_{j+2}$$
$$\vdots$$
$$a_N = \alpha_N - \alpha_{N+1}$$
$$\alpha_j - \alpha_{N+1} = \sum_{n=j}^{N} a_n$$
If $\alpha_{N+1} \xrightarrow[N\to\infty]{} 0$ we get
$$\alpha_j = \sum_{n=j}^{\infty} a_n \qquad \text{and we're done.}$$
We showed
$$0 \leq |a_N - \alpha_N| \leq \|\tilde{a} - \hat{S}_N\|_1 \longrightarrow 0$$
i.e $\lim_{N\to\infty} (a_N - \alpha_N) = 0$

But $\sum_{j\in\mathbb{N}} |a_j| < \infty \Rightarrow a_N \xrightarrow[N\to\infty]{} 0$

and we conclude that

$$\alpha_N = \alpha_N - a_N + a_N \xrightarrow[N\to\infty]{} 0 .$$