Problem Class 3

Exercise 1. Consider the space $\ell_p(\mathbb{N})$ for some $1 \le p < \infty$. Show that $\ell_p(\mathbb{N}) \subset_{\neq} \ell_\infty(\mathbb{N})$ and conclude that the norm of $\ell_\infty(\mathbb{N})$, $\|\cdot\|_\infty$, is a norm on $\ell_p(\mathbb{N})$. Show that this norm is not equivalent to the standard norm $\|\cdot\|_p$.

Solution: If $\bar{a} \in \ell_p(\mathbb{N})$ for $1 \le p < \infty$

then $\sum_{n\in\mathbb{N}} |a_n|^p < \infty$

$\rightarrow$ $a_n \xrightarrow[n\to\infty]{} 0$ and consequently

$\{a_n\}_{n\in\mathbb{N}}$ is bounded, i.e in $\ell_\infty(\mathbb{N})$.

Note that

$$\vec{a} = (1, 1, \dots, 1, \dots) \in \ell_\infty(\mathbb{N})$$

but $\bar{a} \notin \ell_p(\mathbb{N})$ for any $1 \le p < \infty$.

To explore the equivalence of the norms we notice that

$\forall n \in \mathbb{N}$ $\quad |a_n| \le \left(\sum_{k\in\mathbb{N}} |a_k|^p\right)^{1/p} = \|\bar{a}\|_p$

$\Rightarrow$

$$\|\bar{a}\|_\infty = \sup_{n\in\mathbb{N}} |a_n| \le \|\bar{a}\|_p$$

Consider

$$\vec{a}_n = (1, 1, \ldots, \underset{n\text{-th position}}{\widetilde{1}}, 0, 0, \ldots)$$

$$\|\vec{a}_n\|_\infty = 1$$

$$\|\vec{a}_n\|_p = n^{1/p}$$

Had there been $C>0$ s.t

$$\|\vec{x}\|_p \leq C \|\vec{x}\|_\infty \quad \forall x \in \ell_p(\mathbb{N})$$

then

$$n^{1/p} = \|\vec{a}_n\|_p \leq C \|\vec{a}_n\|_\infty = C$$

i.e

$$C \geq n^{1/p} \quad \forall n \in \mathbb{N}$$

which is impossible!

$\Rightarrow$ The norms are <u>not</u> equivalent.

Exercise 2. The goal of this exercise is to prove the following theorem by F. Riesz: Let $\mathscr{X}$ be a normed space and let $\mathscr{M}$ be a closed subspace of $\mathscr{X}$. If $\mathscr{M} \neq \mathscr{X}$ then for any $\varepsilon \in (0,1)$ there exists $x \in \mathscr{X}$ of norm 1 such that

$$\inf_{y\in\mathscr{M}} \|x-y\| \geq 1-\varepsilon.$$

If $\mathscr{M}$ is finite dimensional the above can be improved to

$$\inf_{y\in\mathscr{M}} \|x-y\| \geq 1.$$

(i) For any $z \notin \mathscr{M}$ show that $d_z = \inf_{y\in\mathscr{M}} \|z-y\| > 0$.

(ii) Choosing an arbitrary such $z \notin \mathscr{M}$ find $y_\varepsilon \in \mathscr{M}$ such that $\|z-y_\varepsilon\| \leq (1+\varepsilon)\, d_z$ and show that $x = \frac{z-y_\varepsilon}{\|z-y_\varepsilon\|}$ satisfies

$$\inf_{y\in\mathscr{M}} \|x-y\| \geq 1-\varepsilon.$$

(iii) If $\mathscr{M}$ is finite dimensional show that you can find $y_* \in \mathscr{M}$ such that $\|z-y_*\| = d_z$ and conclude the improved result.

Solution:

(i) $d_z = \inf_{y\in\mathcal{M}} \|z-y\|$

If $d_z = 0$ then $\exists \{y_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$

s.t $\|y_n - z\| < d_z + 1/n = 1/n$

$\Rightarrow y_n \xrightarrow[n\to\infty]{} z$. Since $\mathcal{M}$ is closed

$z \in \mathcal{M}$, a contradiction.

(ii) Let $y_\varepsilon \in \mathcal{M}$ be s.t

$\|z-y_\varepsilon\| < (1+\varepsilon) d_z$

$x = \dfrac{z-y_\varepsilon}{\|z-y_\varepsilon\|}$

For any $y \in \mathcal{M}$

$$\|x - y\| = \left\| \frac{z - y_\varepsilon}{\|z - y_\varepsilon\|} - y \right\|$$

$$= \frac{1}{\|z - y_\varepsilon\|} \left\| z - y_\varepsilon - y\|z - y_\varepsilon\| \right\|$$

$$= \frac{1}{\|z - y_\varepsilon\|} \| z - (\underbrace{y_\varepsilon + \|z - y_\varepsilon\| y}_{\text{in } \mathcal{M} \text{ since } \mathcal{M} \text{ is a subspace}}) \|$$

$$\geq \frac{\inf_{\tilde{y} \in \mathcal{M}} \|z - \tilde{y}\|}{\|z - y_\varepsilon\|} = \frac{d_z}{\|z - y_\varepsilon\|}$$

$$\geq \frac{\cancel{d_z}}{(1+\varepsilon)\cancel{d_z}} = \frac{1}{1+\varepsilon} \geq 1 - \varepsilon$$

$\|x\| = 1$ and we're done.

(iii) Let $\{y_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ be s.t.

$$\|z - y_n\| \xrightarrow[n \to \infty]{} d_z$$

we find that

$$\|y_n\| \leq \|z - y_n\| + \|z\|$$

$$< \sup_{n \in \mathbb{N}} \|z - y_n\| + \|z\| = C < \infty$$

$\|z - y_n\| \to 0$

$\{y_n\}_{n\in\mathbb{N}} \subset B_C(0) \cap \mathcal{M} = \{x \in \mathcal{M} \mid \|x\| \leq C\}$

This is compact since $\mathcal{M}$ is finite dim.

Let $\{e_1, \dots, e_n\}$ be a basis for $\mathcal{M}$.

Any $x \in \mathcal{M}$ can be written uniquely as
$$x = \sum_{i=1}^{n} \alpha_i(x) e_i$$
for unique $\alpha_1(x), \dots, \alpha_n(x) \in \mathbb{F}$.

Define
$$\|x\|_{Euclid} = \sqrt{\sum_{i=1}^{n} |\alpha_i(x)|^2}$$

It is a norm on $\mathcal{M}$.

Consider $I : (\mathbb{F}^n, \|\cdot\|_2) \to (\mathcal{M}, \|\cdot\|_{Euclid})$
$$I(\alpha_1, \dots \alpha_n) = \sum_{i=1}^{n} \alpha_i e_i$$

(*) $\|I(\alpha_1, \dots, \alpha_n) - I(\beta_1, \dots \beta_n)\|_{Eucl} = \|\vec{\alpha} - \vec{\beta}\|_2$

$\Rightarrow$ I is injective.

I is also surjective $(I^{-1}x = (\alpha_1(x), \dots, \alpha_n(x)))$

(*) shows that I is cont. (also I^{-1})

$\Rightarrow$
$$I(\underbrace{|\vec{\alpha}| \leq C}_{\text{compact}}) = \{x \in \mathcal{M} \mid \|x\|_{Eucl\text{-}d} \leq C\}$$

$\Rightarrow$ $\{x \in \mathcal{M} \mid \|x\| \leq C\}$ is compact.

i.e. if $\{y_n\}_{n\in\mathbb{N}}$ is bounded in

$\|\cdot\|_{\text{Euclid}}$ it has a converging subsequence. <u>But</u> since all norms on $\mathcal{U}$ are equivalent, if $\{y_n\}_{n\in\mathbb{N}}$ is bounded in <u>any</u> norm it has a converging subsequence.

$\Rightarrow \{x \in \mathcal{U} \mid \|x\| \le C\}$ is compact.

Back to our problem

$$\|y_n\| \le C \quad \forall n$$

$$\|z - y_n\| \xrightarrow[n\to\infty]{} d_z$$

We find subsequence of $\{y_n\}_{n\in\mathbb{N}}$, $\{y_{n_k}\}_{k\in\mathbb{N}}$, that converges to some element y_0 and then

$$\|z - y_0\| = \lim_{n\to\infty} \|z - y_n\| = d_z$$

Now repeat (ii) with y_0

Exercise 3. Using F. Riesz's theorem show the following: Let $\mathscr{X}$ be an infinite dimensional Banach space. Then there exists sequence $\{x_n\}_{n\in\mathbb{N}}$ of norm 1 such that for all $n \neq m \in \mathbb{N}$

$$\|x_n - x_m\| \geq 1.$$

Consequently $\overline{B}_M(0)$ is not compact for any $M > 0$.

Solution in typed notes.

Exercise 4. Consider the integration operator $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ defined by

$$Tf(x) = \int_a^x f(t)dt.$$

Show that T is bounded.

Solution: Tf is C^1 by the fundamental theorem of calculus and as such cont.

We need to show that there exists $C>0$ s.t

$$\|Tf\|_\infty \leq C\|f\|_\infty \quad \forall f$$

$\forall x \in [a,b]$

$$|Tf(x)| = \left|\int_a^x f(t)dt\right|$$

$$\leq \int_a^x |f(t)|dt \leq \int_a^x \|f\|_\infty dt$$

$$= (x-a)\|f\|_\infty$$

$$\sup_{x\in[a,b]} |Tf(x)| \leq \sup_{x\in[a,b]} (x-a)\|f\|_\infty$$

$$= (b-a)\|f\|_\infty \quad \blacksquare$$