#Functional

Problem class 4

Exercise 1. Prove the following statement: Let $\mathscr{X}$ be a Banach space with Schauder basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$ and denote by

$$\mathscr{X}_n = \overline{\operatorname{span}\{e_k\}_{k\in\mathbb{N},\, k\neq n}}.$$

If for every $n \in \mathbb{N}$ we have that $e_n \notin \mathscr{X}_n$ then there exists a unique sequence $\{f^{(n)}\}_{n\in\mathbb{N}} \subset \mathscr{X}^*$ such that $f^{(n)}(e_j) = \delta_{n,j}$. Moreover, denoting by

$$d_n = \inf_{y\in\mathscr{X}_n} \|e_n - y\|$$

we have that $d_n > 0$ for every $n \in \mathbb{N}$ and $\|f^{(n)}\| = \frac{1}{d_n}$.

__Sketch:__ Since X_n is closed and $e_n \notin X_n$ we have that

$$d_n = \inf_{y\in X_n} \| e_n - y \| > 0.$$

Let's construct $f^{(n)}$. Any $x \in X$ can be written uniquely as

$$x = \sum_{j\in\mathbb{N}} \alpha_j(x)\, e_j$$

If $f^{(n)}(e_j) = \delta_{n,j}$ __and__ $f^{(n)} \in X^*$ then

$$f^{(n)}(x) = f^{(n)}\Big(\sum_{j\in\mathbb{N}} \alpha_j(x)\, e_j\Big)$$

$$= \sum_{j \in \mathbb{N}} \alpha_j(x) f^{(n)}(e_j) = \alpha_n(x)$$

cont.

Define $f^{(n)}(x) = \alpha_n(x)$. We'll show that it is bounded

$$\|x\| = \Big\| \sum_{j \in \mathbb{N}} \alpha_j(x) e_j \Big\| = \Big\| \alpha_n(x) e_n + \sum_{\substack{j \in \mathbb{N} \\ j \neq n}} \alpha_j(x) e_j \Big\|$$

if $\alpha_n(x) \neq 0$

$$= |\alpha_n(x)| \, \Big\| e_n + \frac{\sum_{\substack{j \in \mathbb{N} \\ j \neq n}} \alpha_j(x) e_j}{\alpha_n(x)} \Big\|$$

$$= |f_n(x)| \, \Big\| e_n - \underbrace{\qquad\qquad}_{\text{in } X_n} \Big\|$$

$$\geq |f_n(x)| \, d_n$$

$$\Rightarrow \quad |f^{(n)}(x)| \leq \frac{\|x\|}{d_n} \quad \text{when } f^{(n)}(x) \neq 0.$$

Rest - in typed notes.

Exercise 2. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. A linear operator T is called *compact* if for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathscr{X}$ there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \in \mathscr{X}$ such that the sequence $\{Tx_{n_k}\}_{k\in\mathbb{N}} \in \mathscr{Y}$ converges. Show that if T is compact then for any bounded set $M \subseteq X$ we have that $\overline{T(M)}$ is compact in $\mathscr{Y}$ and conclude that if T is compact then it is bounded.

Hint: You may use the following known fact from the theory of metric spaces: A set A in a metric space satisfies that $\overline{A}$ is compact[1] *if and only if any sequence of elements in A has a subsequence that converges in the space.*

Solution: Let $\{y_n\}_{n\in\mathbb{N}} \subset T(M)$

we'd like to show that there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}} \subset T(M)$ that converges (in $\mathscr{Y}$).

By definition $\exists \{x_n\}_{n\in\mathbb{N}} \subset M$ s.t

$$y_n = Tx_n$$

$\{x_n\}_{n\in\mathbb{N}}$ is bounded as it is in the bounded set M, and as T is compact, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ s.t

$$y_{n_k} = Tx_{n_k}$$

converges!

$\Rightarrow \overline{T(M)}$ is compact.

To show continuity we notice that

$$M = \{x \in X \mid \|x\| = 1\}$$

is bounded.

$\Rightarrow \overline{T(M)}$ is compact and as such bounded

$\Rightarrow \exists C > 0$ s.t $\forall x \in M$

$$\underbrace{\|Tx\|}_{\in T(M)} \leq C$$

$\Rightarrow$

$$\sup_{\|x\|=1} \|Tx\| \leq C$$

i.e T is bounded.

Exercise 3. Show that there exists a functional in $\ell_\infty(\mathbb{N})^*$ that is not of the form $f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$.
Hint: Using the fact that

$$f_{\sum_{n=1}^{N} \alpha_n \boldsymbol{e}_n} = \sum_{n=1}^{N} \overline{\alpha_n} f_{\boldsymbol{e}_n}$$

conclude that if every $\ell_\infty(\mathbb{N})^*$ *is of the form* $f_{\boldsymbol{b}}$ *for some* $\boldsymbol{b} \in \ell_1(\mathbb{N})$ *then* $\ell_\infty(\mathbb{N})^*$ *is separable.*

Solution: We know that for any $\vec{b} \in \ell_1(\mathbb{N})$

$f_{\vec{b}}$ (defined as

$$f_{\vec{b}}(\vec{a}) = \sum a_n b_n \quad)$$

is well defined and $\|f_{\vec{b}}\| \leq \|\vec{b}\|_1$.

We claim that

$$f_{\sum_{n=1}^{N} \alpha_n \vec{e}_n} = \sum_{n=1}^{N} \overline{\alpha_n} f_{\vec{e}_n}$$

Indeed for any $\vec{a} \in \ell_\infty(\mathbb{N})$

$$f_{\sum_{n=1}^{N} \alpha_n e_n}(\vec{a}) = \sum_{n\in\mathbb{N}} a_n \left(\sum_{j=1}^{N} \overline{\alpha_j \vec{e}_j}\right)_n$$

$$= \sum_{n=1}^{N} a_n \overline{\alpha_n}$$

$$\sum_{n=1}^{N} \overline{\alpha_n} f_{e_n}(\vec{a}) = \sum_{n=1}^{N} \overline{\alpha_n} \left(\sum_{j\in\mathbb{N}} a_j \overline{(\vec{e}_n)_j}\right)$$

$$= \sum_{n=1}^{N} a_n \overline{\alpha_n}$$

We know that $\{\vec{e}_n\}_{n\in\mathbb{N}}$ is a Schauder basis to $\ell_1(\mathbb{N}) \Rightarrow \forall \vec{b}\in\ell_1(\mathbb{N})$ and any $\varepsilon>0$ $\exists N$ and $a_1,\dots,a_N\in\mathbb{F}$

$$\|\vec{b}-\sum_{n=1}^{N} a_n e_n\|_1<\varepsilon$$

$$\Rightarrow \|f_b-\sum_{n=1}^{N}\bar{a}_n f_{e_n}\| = \|f_b - f_{\sum a_n e_n}\|$$

$$= \|f_{\vec{b}-\sum_{n=1}^{N} a_n \vec{e}_n}\| \leq \|\vec{b}-\sum_{n=1}^{N} a_n e_n\|_1 < \varepsilon$$

$\Rightarrow$ $\operatorname{span}\{f_{e_n}\}$ is dense in the set $\{f_{\vec{b}} \mid \vec{b}\in\ell_1(\mathbb{N})\}$

Thus if $\ell_\infty(\mathbb{N})^*$ consists only of $f_{\vec{b}}$-s with $\vec{b}\in\ell_1(\mathbb{N})$ we find that

$$\ell_\infty(\mathbb{N})^* = \overline{\operatorname{span}\{f_{e_n}\}} \rightarrow \text{separable}$$

$\rightarrow$ $\ell_\infty(\mathbb{N})^*$ is separable, but according to a theorem from class this implies that $\ell_\infty(\mathbb{N})$ is separable, a contradiction.

Exercise 4. Let $\mathscr{H}$ be a Hilbert space. Show that the following conditions are equivalent:

(i) $x_n \xrightarrow[n\to\infty]{} x$.

(ii) $x_n \xrightarrow[n\to\infty]{w} x$ and $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$.

Solution:

we'll show that (i) $\Rightarrow$ (ii) in a general Banach space.

If $x_n \xrightarrow[n\to\infty]{} x$ (i.e $\|x_n - x\| \longrightarrow 0$)

we know that

$$\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$$

by the continuity of the norm.

Next consider $f \in X^*$

$$0 \le |f(x_n) - f(x)| = |f(x_n - x)|$$

$$\le \|f\| \, \|x_n - x\|$$

$$\downarrow n\to\infty$$

$$0$$

By pinching lemma $|f(x_n) - f(x)| \xrightarrow[n\to\infty]{} 0$

i.e $f(x_n) \xrightarrow[n\to\infty]{} f(x)$.

Note: f is cont. so we could have immediately said that

$$f(x_n) \xrightarrow[n\to\infty]{} f(x)$$

(ii) $\Rightarrow$ (i) relies on the Hilbert structure.

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\mathrm{Re}\langle x_n, x\rangle + \|x\|^2$$

We know that for any $y \in H$

$$f_y(x) = \langle x, y\rangle$$

is in H^*. As $x_n \xrightarrow{w} x$

$$\langle x_n, x\rangle = f_x(x_n) \xrightarrow[n\to\infty]{} f_x(x) = \langle x, x\rangle = \|x\|^2$$

Also $\|x_n\|^2 \xrightarrow[n\to\infty]{} \|x\|^2$

$\Rightarrow$

$$\|x_n - x\|^2 \xrightarrow[n\to\infty]{} \|x\|^2 - 2\mathrm{Re}\,\|x\|^2 + \|x\|^2 = 0$$