Problem Class 1

Exercise 1. Consider the space $(C[0,1], \|\cdot\|_\infty)$ and the sets

$$C^1[0,1] = \left\{ f : [0,1] \to \mathbb{R} \,\middle|\, f \text{ is continuously differentiable on } [0,1] \right\},$$

$$C_0^1[0,1] = \left\{ f : [0,1] \to \mathbb{R} \,\middle|\, f \text{ is continuously differentiable on } [0,1],\ f(0) = f(1) = 0 \right\}.$$

(i) Show that $C^1[0,1]$ and $C_0^1[0,1]$ are subspaces of $C[0,1]$.
(ii) Show that $C_0^1[0,1]$ is a subspace of $C^1[0,1]$.

Define a function $\|\cdot\|_{C^1} : C^1[0,1] \to \mathbb{R}_+$ by

$$\|f\|_{C^1} = \|f'\|_\infty .$$

(iii) Show that $\|\cdot\|_{C^1}$ is not a norm on $C^1[0,1]$ but is a norm on $C_0^1[0,1]$.
(iv) Show that $\left(C_0^1[0,1], \|\cdot\|_{C^1}\right)$ is a Banach space.

Exercise 2. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space.

(i) Show that if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{X}$ then one can extract a subsequence of it, $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$ such that

$$\left\| x_{n_k} - x_{n_{k+1}} \right\| < \frac{1}{2^k}.$$

We say that a series $\sum_{n\in\mathbb{N}} x_n$ *converges* in $\mathscr{X}$ is the sequence of partial sums, $\{S_N\}_{N\in\mathbb{N}}$, defined as

$$S_N = \sum_{n=1}^{N} x_n$$

converges in $\mathscr{X}$. We say that a series $\sum_{n\in\mathbb{N}} x_n$ *converges absolutely* in $\mathscr{X}$ if $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.

(ii) Show that if $(\mathscr{X}, \|\cdot\|)$ is a Banach space then every absolutely converging series converges.
(iii) Show that if $(\mathscr{X}, \|\cdot\|)$ is a normed space where every absolutely converging series converges, then $(\mathscr{X}, \|\cdot\|)$ is a Banach space.
Hint: You may use the fact that any Cauchy sequence of a converging subsequence converges to the same limit as the original sequence.

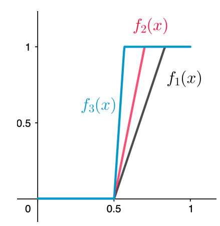
Exercise 3 *(If time permits)*. Consider the vector space $C[0,1]$ and define the function $\|\cdot\|_1 : C[0,1] \to \mathbb{R}_+$ by

$$\|f\|_1 = \int_0^1 \left| f(x) \right| dx.$$

You may assume without proof that $(C[0,1], \|\cdot\|_1)$ is a normed space. Show that it is not a Banach space.

Hint: Consider the sequence

$$f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ n\left(x - \frac{1}{2}\right) & \frac{1}{2} \leq x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$



and use the fact that for any $0 \leq a \leq b \leq 1$ *we have that for any* $g \in C[0,1]$

$$\int_a^b |g(x)|\, dx \leq \int_0^1 |g(x)|\, dx.$$