Problem Class 1 Solution

Solution to Question 1. We start by noticing that the zero function, $\mathbf{0}$, belongs to $C^1[0,1]$ and $C_0^1[0,1]$.
Next we use the fact that pointwise addition and pointwise multiplication of continuously differentiable functions is continuously differentiable to conclude that $C^1[a,b]$ is closed under the addition and scalar multiplication operations. This shows that $C^1[0,1]$ is indeed a subspace of $C[0,1]$. Moreover, if $f,g \in C_0^1[0,1]$ then

$$\left(f+g\right)(0) = f(0)+g(0) = 0+0 = 0, \qquad \left(f+g\right)(1) = f(1)+g(1) = 0+0 = 0$$

which shows that $f+g \in C_0^1[0,1]$, and similarly for any $f \in C^1[0,1]$ and a scalar α

$$\left(\alpha f\right)(0) = \alpha f(0) = 0, \qquad \left(\alpha f\right)(1) = \alpha f(1) = 0$$

which shows that $\alpha f \in C_0^1[0,1]$. We conclude that $C_0^1[0,1]$ is also a subspace.

(iii) We notice that

$$\left\|f\right\|_{C^1} = 0 \;\Leftrightarrow\; \left\|f'\right\|_\infty = 0 \;\Leftrightarrow\; f' = 0 \;\forall x \in (0,1) \;\Leftrightarrow\; f \equiv \text{Const.}.$$

As any constant is in $C^1[0,1]$, we see that $\left\|f\right\|_{C^1} = 0$ *doesn't* imply that $f = \mathbf{0}$ in $C^1[0,1]$. For instance, consider $f \equiv 1$. This shows that $\|\cdot\|_{C^1}$ is not a norm on $C^1[0,1]$. If, however, $f \in C_0^1[0,1]$ then since $f(0) = 0$ we conclude that if f is constant, then $f = \mathbf{0}$. We'll continue to check the scalar property and triangle inequality on $C_0^1[0,1]$.
For any $f \in C_0^1[0,1]$ and any scalar α we know from basic rules of differentiation and the fact that $\|\cdot\|_\infty$ is a norm that

$$\left\|\alpha f\right\|_{C^1} = \left\|\left(\alpha f\right)'\right\|_\infty = \left\|\alpha f'\right\|_\infty = |\alpha| \left\|f'\right\|_\infty = |\alpha| \left\|f\right\|_{C^1}.$$

Similarly, for any $f,g \in C_0^1[0,1]$

$$\left\|f+g\right\|_{C^1} = \left\|\left(f+g\right)'\right\|_\infty = \left\|f'+g'\right\|_\infty \le \left\|f'\right\|_\infty + \left\|g'\right\|_\infty = \left\|f\right\|_{C^1} + \left\|g\right\|_{C^1},$$

which shows the triangle inequality. We conclude that $\|\cdot\|_{C^1}$ is indeed a norm on $C_0^1[0,1]$.

(iv) We start by noticing that $\left\{f_n\right\}_{n\in\mathbb{N}} \in C_0^1[0,1]$ is Cauchy in $\left(C_0^1[0,1], \|\cdot\|_{C^1}\right)$ implies (by definition) that $\left\{f_n'\right\}_{n\in\mathbb{N}} \in C[0,1]$ is Cauchy. We know from class that $C[0,1]$ is complete, which implies that there exists $g \in C[0,1]$ such that $\lim_{n\to\infty} \left\|f_n' - g\right\|_\infty = 0$. g is our candidate for *the derivative* of our limit function f. We thus define

$$f(x) = \int_0^x g(y)dy.$$

By the fundamental theorem of Calculus we have that $f \in C^1[0,1]$ with $f' = g$ and $f(0) = 0$. Moreover, if $f(1) = 0$ we'll conclude that $f \in C^1[0,1]$ and

$$\|f_n - f\|_{C^1} = \|f_n' - g\|_\infty \underset{n\to\infty}{\longrightarrow} 0,$$

which will show the completeness of $\left(C_0^1[0,1], \|\cdot\|_{C^1}\right)$. Indeed, since $\{f_n'\}_{n\in\mathbb{N}} \in C_0^1[0,1]$ converges uniformly to g we have that

$$f(1) = \int_0^1 g(y)dy = \int_0^1 \left(\lim_{n\to\infty} f_n'(y)\right)dy = \lim_{n\to\infty}\int_0^1 f_n'(y)dy$$
$$= \lim_{n\to\infty}\left(f_n(1) - f_n(0)\right) = \lim_{n\to\infty}(0) = 0.$$

The proof is thus complete.

Solution to Question 2. (i) The claim is actually true in general metric spaces. In that case it reads as: if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X then one can extract a subsequence of it, $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$d\left(x_{n_k}, x_{n_{k+1}}\right) < \frac{1}{2^k}.$$

We will prove this more general statement as the proof of it and (**??**) is identical.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. For $\varepsilon_1 = \frac{1}{2}$ we can find $n_1 \in \mathbb{N}$ such that for all $n, m \geq n_1$ we have that

$$d(x_n, x_m) < \frac{1}{2}.$$

For $\varepsilon_2 = \frac{1}{4}$ we can find $\widetilde{n}_2 \in \mathbb{N}$ such that for all $n, m \geq \widetilde{n}_2$ we have that

$$d(x_n, x_m) < \frac{1}{4}.$$

Defining $n_2 = \max(n_1 + 1, \widetilde{n}_2)$ we find that $n_2 > n_1$ and for any $n, m \geq n_2$ we have that

$$d(x_n, x_m) < \frac{1}{4}.$$

We continue by induction. Assume that for a fixed $k \in \mathbb{N}$ we have found $n_k > n_{k-1}$ such that for all $n, m > n_k$ we have that

$$d(x_n, x_m) < \frac{1}{2^k}.$$

For $\varepsilon_{k+1} = \frac{1}{2^{k+1}}$ we can find $\widetilde{n}_{k+1} \in \mathbb{N}$ such that for all $n, m \geq \widetilde{n}_{k+1}$ we have that

$$d(x_n, x_m) < \frac{1}{2^{k+1}}.$$

Defining $n_{k+1} = \max(n_k + 1, \widetilde{n}_{k+1})$ we find that $n_{k+1} > n_k$ and for any $n, m \geq n_{k+1}$ we have that

$$d(x_n, x_m) < \frac{1}{2^{k+1}}.$$

The set $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ (since $n_{k+1} > n_k$ for any $k \in \mathbb{N}$) and for any $k \in \mathbb{N}$ we have that $n_k, n_{k+1} \geq n_k$ and as such

$$d\left(x_{n_k}, x_{n_{k+1}}\right) < \frac{1}{2^k}.$$

The result is thus shown.

(ii) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ such that $\sum_{n\in\mathbb{N}} x_n$ converges absolutely and consider the partial sums sequence $\{S_N\}_{N\in\mathbb{N}}$. Since $\mathcal{X}$ is complete it is enough for us to show that $\{S_N\}_{N\in\mathbb{N}}$ is Cauchy to know that it converges. Indeed

$$\|S_N - S_M\| = \left\| \sum_{\min\{N,M\}+1}^{\max\{N,M\}} x_n \right\| \leq \sum_{\min\{N,M\}+1}^{\max\{N,M\}} \|x_n\| = |s_N - s_M|$$

where $s_N = \sum_{n=1}^{N} \|x_n\|$. Since the series converges absolutely we know that $\{s_N\}_{N\in\mathbb{N}}$ converges and as such Cauchy. Thus, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $N, M \geq n_0$ we have that $|s_N - s_M| < \varepsilon$. Consequently, if $N, M \geq n_0$

$$\|S_N - S_M\| < \varepsilon,$$

which shows the desired Cauchy criterion. The proof is concluded.

(iii) We assume that $\mathcal{X}$ is a normed space where every absolutely convergent series also converges and we'll show that any Cauchy sequence in $\mathcal{X}$ has a limit. This follows from the following:

- Given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ we can use the first part of the exercise to find a subsequence of it, $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\left\|x_{n_k} - x_{n_{k+1}}\right\| < \frac{1}{2^k}.$$

- Since $\sum_{k\in\mathbb{N}} \left\|x_{n_k} - x_{n_{k+1}}\right\| < 1$ we conclude, from our assumption, that

$$S_N = \sum_{k=1}^{N} \left(x_{n_{k+1}} - x_{n_k}\right)$$

converges to some vector $S \in \mathcal{X}$.

- As
$$S_N = x_{n_{N+1}} - \underbrace{x_{n_1}}_{\substack{\text{fixed}\\ \text{vector}}}$$
we conclude that $\left\{x_{n_{N+1}}\right\}_{N\in\mathbb{N}}$ converges to $S - x_{n_1}$.
- As we have found a subsequence to $\{x_n\}_{n\in\mathbb{N}}$, which is a Cauchy sequence, that converges we can conclude that $\{x_n\}_{n\in\mathbb{N}}$ must also converge.

Since $\{x_n\}_{n\in\mathbb{N}}$ was arbitrary we have shown that any Cauchy sequence converges. Thus $\mathscr{X}$ is a Banach space.

Solution to Question 3. Assuming that $n > m$ without loss of generality we see that
$$\left\|f_n - f_m\right\|_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} \left|f_n(x) - f_m(x)\right| dx \leq \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} 2dx = \frac{2}{m},$$
where we have used the fact that $0 \leq f_n(x) \leq 1$ for any $x \in [0,1]$ and any $n \in \mathbb{N}$. We conclude that
$$\left\|f_n - f_m\right\|_1 < \varepsilon$$
when $n, m > \frac{2}{\varepsilon}$, which shows that $\left\{f_n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\|\cdot\|_1$.
We now need to show that f_n doesn't converge to any function in $C[0,1]$. We can immediately see that f_n converges pointwise to a function $\widetilde{f}$ such that
$$\widetilde{f}(x) = 0, \ \ x \in \left[0, \frac{1}{2}\right] \quad \text{and} \quad \widetilde{f}(x) = 1 \ \ x \in \left(\frac{1}{2}, 1\right].$$
Looking at the left and right limits at $x = \frac{1}{2}$ we conclude that $\widetilde{f}$ is not continuous. Intuitively speaking, we expect that if f_n converges to anything in $\|\cdot\|_1$, then it should be the above function[1]. We will proceed by assuming that $\left\{f_n\right\}_{n\in\mathbb{N}}$ converges to a function $f \in C[0,1]$ and prove that f can't be continuous, showing the desired contradiction.
Assume that $\left\{f_n\right\}_{n\in\mathbb{N}}$ converges to $f \in C[0,1]$. Then, since
$$\int_0^{\frac{1}{2}} \left|f(x)\right| dx = \int_0^{\frac{1}{2}} \left|f(x) - f_n(x)\right| dx \leq \int_0^1 \left|f(x) - f_n(x)\right| dx = \left\|f - f_n\right\|_1$$
for all $n \in \mathbb{N}$, we see that by taking n to infinity we find that
$$0 \leq \int_0^{\frac{1}{2}} \left|f(x)\right| dx \leq \lim_{n\to\infty} \left\|f - f_n\right\|_1 = 0.$$

[1]This, in fact, is true in any of the L^p norms. A known theorem in Measure Theory states that if $\left\{f_n\right\}_{n\in\mathbb{N}}$ converges to f in $\|\cdot\|_p$ then it has a subsequence, $\left\{f_{n_k}\right\}_{k\in\mathbb{N}}$, that converges pointwise almost everywhere to f.

This implies that $\int_0^{\frac{1}{2}} \left|f(x)\right| dx = 0$ and since f is continuous we conclude that $f(x) = 0$ on $\left[0, \frac{1}{2}\right]$. Similarly, for any $\delta > 0$ we have that if $n > \frac{1}{\delta}$ then[2]

$$\int_{\frac{1}{2}+\delta}^{1} \left|f(x) - 1\right| dx = \int_{\frac{1}{2}+\delta}^{1} \left|f(x) - f_n(x)\right| dx \leq \left\|f - f_n\right\|_1,$$

from which we conclude that $f(x) = 1$ on $\left[\frac{1}{2} + \delta, 1\right]$ for any $\delta > 0$. Thus, as δ was arbitrary, $f(x) = 1$ on $\left(\frac{1}{2}, 1\right]$. Noticing that

$$\lim_{x \to \frac{1}{2}^-} f(x) = 0 \neq 1 = \lim_{x \to \frac{1}{2}^+} f(x)$$

we conclude that f can't be continuous - contradicting the convergence in $C[0,1]$. We conclude that $\left\{f_n\right\}_{n \in \mathbb{N}}$ has no limit and as such $(C[0,1], \|\cdot\|_1)$ is not complete.
In almost the exact same way we could have shown that $\left(C[0,1], \|\cdot\|_p\right)$, where $\|\cdot\|_p$ is the norm defined on $L^p[0,1]$, is not complete for any $1 < p < \infty$.

[2] As $n > \frac{1}{\delta}$ we have that $\frac{1}{2} + \frac{1}{n} < \frac{1}{2} + \delta$.