Problem Class 2 Solution

Solution to Question 1. Assume by contradiction that $\mathscr{B}$ is a Schauder basis. Then for any $f \in C[a,b]$ there exists a sequence of scalars $\{\alpha_n(f)\}_{n\in\mathbb{N}}$ such that the sequence of functions

$$S_N(x) = \sum_{n=0}^{N} \alpha_n(f) x^n$$

converges to f in the norm $\|\cdot\|_\infty$, i.e. uniformly. This, however, implies $f(x) = \sum_{n\in\mathbb{N}} \alpha_n(f) x^n$, which means that f has a *power series* and thus is analytic on (a,b). Since not every continuous function is analytic, we have reached the desired contradiction.

Solution to Question 2. Let $\mathscr{B} = \{e_\alpha\}_{\alpha\in\mathscr{I}}$ be an uncountable orthonormal set. Then, for any $\alpha \neq \beta$ we have that

$$\|e_\alpha - e_\beta\| = \sqrt{\|e_\alpha\|^2 - 2\mathrm{Re}\langle e_\alpha, e_\beta\rangle + \|e_\beta\|^2} = \sqrt{2}.$$

According to a theorem in class, since we found an uncountable set of elements such that $d(e_\alpha, e_\beta) > \eta > 0$ for $\eta = \sqrt{2}$ the space can't be separable.

Solution to Question 3. It is relatively straightforward to show that $\widetilde{\mathscr{B}}$ is independent. Indeed, it is enough to show that $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k\}$ are independent for any $k \in \mathbb{N}$. Assume that

$$\sum_{j=1}^{k} \alpha_j \boldsymbol{x}_j = 0.$$

Then, by definition,

$$0 = \alpha_1 \boldsymbol{e}_1 + \sum_{j=2}^{k} \alpha_j (\boldsymbol{e}_j - \boldsymbol{e}_{j-1}) = \sum_{j=1}^{k-1} (\alpha_j - \alpha_{j+1}) \boldsymbol{e}_j + \alpha_k \boldsymbol{e}_k.$$

Since $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ are independent we have that $\alpha_k = 0$ and $\alpha_j = \alpha_{j+1}$ for any $j = 1, \dots, k-1$. This implies that $\alpha_j = 0$ for any $j = 1, \dots, k$ which concludes the independence.
Next we notice that

$$\sum_{j=1}^{n} \boldsymbol{x}_j = \boldsymbol{e}_1 + \sum_{j=2}^{n} (\boldsymbol{e}_j - \boldsymbol{e}_{j-1}) = \boldsymbol{e}_n.$$

To show the expansion of any element $\boldsymbol{a} \in \ell_1(\mathbb{N})$ with respect to $\widetilde{\mathscr{B}}$ we remind ourselves that for any given $\boldsymbol{a} \in \ell_1(\mathbb{N})$ we have that $S_N = \sum_{n=1}^{N} a_n \boldsymbol{e}_n$

converges to it in norm. Since

$$S_N = \sum_{n=1}^{N} a_n \left(\sum_{j=1}^{n} \boldsymbol{x}_j \right) = \sum_{j=1}^{N} \left(\sum_{n=j}^{N} a_n \right) \boldsymbol{x}_j$$

we would guess an expansion of the form

$$\mathcal{S}_N = \sum_{j=1}^{N} \left(\sum_{n=j}^{\infty} a_n \right) \boldsymbol{x}_j.$$

The above is indeed well defined since $\|a\|_1 = \sum_{n\in\mathbb{N}} |a_n| < \infty$. We find that

$$\mathcal{S}_N = \sum_{j=1}^{N} \left(\sum_{n=j}^{N} a_n + \sum_{n=N+1}^{\infty} a_n \right) \boldsymbol{x}_j = \sum_{j=1}^{N} \left(\sum_{n=j}^{N} a_n \right) \boldsymbol{x}_j + \sum_{j=1}^{N} \left(\sum_{n=N+1}^{\infty} a_n \right) \boldsymbol{x}_j$$

$$= S_N + \left(\sum_{n=N+1}^{\infty} a_n \right) \left(\sum_{j=1}^{N} \boldsymbol{x}_j \right) = S_N + \left(\sum_{n=N+1}^{\infty} a_n \right) \boldsymbol{e}_N.$$

and consequently,

$$\|\mathcal{S}_N - \boldsymbol{a}\|_1 \le \|S_N - \boldsymbol{a}\|_1 + \left| \sum_{n=N+1}^{\infty} a_n \right| \underbrace{\|\boldsymbol{e}_N\|}_{=1} \underset{N\to\infty}{\longrightarrow} 0.$$

To conclude the exercise we need to show that the coefficients in the expansion with respect to $\widetilde{\mathscr{B}}$ are unique. Indeed, if $\widehat{S}_N = \sum_{n=1}^{N} \alpha_n \boldsymbol{x}_n$ converges to $\boldsymbol{a}$ then since

$$\widehat{S}_N = \alpha_1 \boldsymbol{e}_1 + \sum_{n=2}^{N} \alpha_n (\boldsymbol{e}_n - \boldsymbol{e}_{n-1}) = \sum_{n=1}^{N-1} (\alpha_n - \alpha_{n+1}) \boldsymbol{e}_n + \alpha_N \boldsymbol{e}_N$$

and since for a given $j \in \mathbb{N}$ we know (from the home assignment) that

$$\left| a_j - \left(\widehat{S}_N\right)_j \right| \le \left\| \boldsymbol{a} - \widehat{S}_N \right\|.$$

we find that for any $j \le N-1$

$$\left| a_j - \left(\alpha_j - \alpha_{j+1}\right) \right| \le \left\| \boldsymbol{a} - \widehat{S}_N \right\| \tag{1}$$

and for $j = N$ we get that

$$|a_N - \alpha_N| \le \left\| \boldsymbol{a} - \widehat{S}_N \right\| \tag{2}$$

(1) implies that for any fixed j we have that

$$0 \le \left| a_j - \left(\alpha_j - \alpha_{j+1}\right) \right| \le \liminf_{N\to\infty} \left\| \boldsymbol{a} - \widehat{S}_N \right\| = 0$$

i.e.

$$a_j = \alpha_j - \alpha_{j+1}.$$

With this at hand we see that for any fixed $k \in \mathbb{N}$ and $N \geq k$

$$\sum_{j=k}^{N} a_j = \sum_{j=k}^{N} \left(\alpha_j - \alpha_{j+1}\right) = \alpha_k - \alpha_{N+1}.$$

If we'll show that $\lim_{N\to\infty} \alpha_N = 0$ we will conclude that, since $\boldsymbol{a} \in \ell_1(\mathbb{N})$,

$$\alpha_k = \sum_{j=k}^{\infty} a_j,$$

which will show the desired uniqueness of the coefficients.
Indeed, according to (2) we have that

$$0 \leq |a_N - \alpha_N| \leq \left\| \boldsymbol{a} - \widehat{S}_N \right\|$$

which implies that $\lim_{N\to\infty}(a_N - \alpha_N) = 0$. Since $\boldsymbol{a} \in \ell_1(\mathbb{N})$ we know that $\lim_{N\to\infty} a_N = 0$ from which we can now conclude that $\lim_{N\to\infty} \alpha_N = 0$ as well. The proof is thus complete.