Problem Class 3

Exercise 1. Consider the space $\ell_p(\mathbb{N})$ for some $1 \leq p < \infty$. Show that $\ell_p(\mathbb{N}) \subset_{\neq} \ell_\infty(\mathbb{N})$ and conclude that the norm of $\ell_\infty(\mathbb{N})$, $\|\cdot\|_\infty$, is a norm on $\ell_p(\mathbb{N})$. Show that this norm is not equivalent to the standard norm $\|\cdot\|_p$.

Exercise 2. The goal of this exercise is to prove the following theorem by F. Riesz: Let $\mathscr{X}$ be a normed space and let $\mathscr{M}$ be a closed subspace of $\mathscr{X}$. If $\mathscr{M} \neq \mathscr{X}$ then for any $\varepsilon \in (0,1)$ there exists $x \in \mathscr{X}$ of norm 1 such that

$$\inf_{y \in \mathscr{M}} \|x - y\| \geq 1 - \varepsilon.$$

If $\mathscr{M}$ is finite dimensional the above can be improved to

$$\inf_{y \in \mathscr{M}} \|x - y\| \geq 1.$$

(i) For any $z \notin \mathscr{M}$ show that $d_z = \inf_{y \in \mathscr{M}} \|z - y\| > 0$.

(ii) Choosing an arbitrary such $z \notin \mathscr{M}$ find $y_\varepsilon \in \mathscr{M}$ such that $\|z - y_\varepsilon\| \leq (1+\varepsilon)\, d_z$ and show that $x = \frac{z - y_\varepsilon}{\|z - y_\varepsilon\|}$ satisfies

$$\inf_{y \in \mathscr{M}} \|x - y\| \geq 1 - \varepsilon.$$

(iii) If $\mathscr{M}$ is finite dimensional show that you can find $y_* \in \mathscr{M}$ such that $\|z - y_*\| = d_z$ and conclude the improved result.

Exercise 3. Using F. Riesz's theorem show the following: Let $\mathscr{X}$ be an infinite dimensional Banach space. Then there exists sequence $\{x_n\}_{n \in \mathbb{N}}$ of norm 1 such that for all $n \neq m \in \mathbb{N}$

$$\|x_n - x_m\| \geq 1.$$

Consequently $\overline{B}_M(0)$ is not compact for any $M > 0$.

Exercise 4. Consider the integration operator $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ defined by

$$Tf(x) = \int_a^x f(t)\, dt.$$

Show that T is bounded.