Problem Class 3 Solution

Solution to Question 1. We start by noticing that if $\boldsymbol{a} \in \ell_p(\mathbb{N})$ for some $1 \leq p < \infty$ then $\lim_{n\to\infty} a_n = 0$, and as such the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded, i.e. belongs to $\ell_\infty(\mathbb{N})$. Since the addition and scalar multiplication that is defined in all of $\ell_p(\mathbb{N})$ −s, $1 \leq p \leq \infty$, is identical we conclude that $\ell_p(\mathbb{N})$ is a subspace of $\ell_\infty(\mathbb{N})$ and as such we can consider the induced/restricted norm $\|\cdot\|_\infty$ on it.
Next, we notice that for any $n \in \mathbb{N}$

$$|a_n| \leq \left(\sum_{k\in\mathbb{N}} |a_k|^p\right)^{\frac{1}{p}} = \|\boldsymbol{a}\|_p.$$

Taking the supremum over $n \in \mathbb{N}$ gives us

$$\|\boldsymbol{a}\|_\infty \leq \|\boldsymbol{a}\|_p.$$

The converse, however, doesn't hold. Indeed, consider the vectors

$$\boldsymbol{a}_n = \sum_{i=1}^{n} \boldsymbol{e}_i = \Bigg(1, 1, \dots, \underbrace{1}_{n-\text{th position}}, 0, \dots\Bigg).$$

we see that

$$\|\boldsymbol{a}_n\|_\infty = 1$$

$$\|\boldsymbol{a}_n\|_p = n^{\frac{1}{p}},$$

which shows that there can't be a constant $c > 0$ such that

$$\|\boldsymbol{a}\| \leq c\,\|\boldsymbol{a}\|_\infty$$

for all $\boldsymbol{a} \in \ell_p(\mathbb{N})$ since that would have implied that

$$n^{\frac{1}{p}} = \|\boldsymbol{a}_n\|_p \leq c\,\|\boldsymbol{a}_n\|_\infty = c$$

for all $n \in \mathbb{N}$, which is impossible. Consequently the norm $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are not equivalent on $\ell_p(\mathbb{N})$.

Solution to Question 2. (i) By definition, $d_z \geq 0$. If $d_z = 0$ then, by definition, we can find a sequence $\{y_n\}_{n\in\mathbb{N}} \in \mathscr{M}$ such that

$$\|z - y_n\| \leq \inf_{y\in\mathscr{M}} \|z - y\| + \frac{1}{n} = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

This implies that z is a limit of a sequence of elements from $\mathscr{M}$, i.e. z is in $\overline{\mathscr{M}} = \mathscr{M}$, which is a contradiction.

(ii) From the definition of the infimum we see that for any $\varepsilon > 0$ we can find a vector $y_\varepsilon \in \mathscr{M}$ such that

$$d_z \leq \left\| z - y_\varepsilon \right\| \leq (1+\varepsilon)\, d_z.$$

The vector $x_\varepsilon = \frac{z-y_\varepsilon}{\|z-y_\varepsilon\|}$ is of norm 1 and for any $y \in \mathscr{M}$

$$\left\| x - y \right\| = \frac{1}{\left\| z - y_\varepsilon \right\|} \cdot \left\| z - y_\varepsilon - \left\| z - y_\varepsilon \right\| y \right\|$$

$$\underset{y_\varepsilon + \|z-y_\varepsilon\| y \in \mathscr{M}}{\geq} \frac{d_z}{\left\| z - y_\varepsilon \right\|} \geq \frac{1}{1+\varepsilon}.$$

Since $y \in \mathscr{M}$ was arbitrary, this concludes the proof as for any $\varepsilon \in (0,1)$, $\frac{1}{1+\varepsilon} > 1 - \varepsilon$.

(iii) By the definition of the infimum, we can find a sequence $\left\{ y_n \right\}_{n\in\mathbb{N}} \in \mathscr{M}$ such that

$$\left\| z - y_n \right\| \underset{n\to\infty}{\longrightarrow} d_z.$$

Since

$$\left\| y_n \right\| \leq \left\| z - y_n \right\| + \|z\| \leq \underbrace{\sup_{n\in\mathbb{N}} \left\| z - y_n \right\|}_{\text{bounded}} + \|z\|$$

we conclude that $\left\{ y_n \right\}_{n\in\mathbb{N}}$ is a bounded sequence in finite dimensional spaces. As al the norms of finite dimensional spaces are equivalent we see that by choosing the norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |\alpha_i(x)|^2}$$

where $x = \sum_{i=1}^{n} \alpha_i(x) e_i$ for some basis $\{e_1, \dots, e_n\}$ we find an identification (isomorphism) of $(\mathscr{M}, \|\cdot\|_2)$ with $(\mathbb{F}^n, \|\cdot\|_2)$, where $\mathbb{F}$ is the field above which $\mathscr{M}$ acts. As bounded sequences in $\mathbb{F}^n$ must have a converging subsequences, the same holds for bounded sequences in $(\mathscr{M}, \|\cdot\|_2)$ and consequently in any norm on $\mathscr{M}$. Thus, we can find a subsequence of $\left\{ y_n \right\}_{n\in\mathbb{N}}$, $\left\{ y_{n_k} \right\}_{k\in\mathbb{N}}$, that converges to some element y_*. Since $\mathscr{M}$ is finite dimensional it is closed and we find that $y_* \in \mathscr{M}$. Consequently

$$d_z = \lim_{k\to\infty} \left\| z - y_{n_k} \right\| = \left\| z - y_* \right\|.$$

Much like in the proof of the previous part we define $x = \frac{z-y_*}{\|z-y_*\|}$ and find that for any $y \in \mathscr{M}$

$$\left\| x - y \right\| = \frac{1}{\left\| z - y_* \right\|} \cdot \left\| z - y_* - \left\| z - y_* \right\| y \right\| \geq \frac{d}{\left\| z - y_* \right\|} = 1.$$

As y was arbitrary we conclude the result.

Solution to Question 3. Let x_1 be an arbitrary vector in $\mathscr{X}$ with norm 1 and denote by $\mathscr{M}_1 = \text{span}\{x_1\}$. According to F. Riesz's theorem we can find x_2 of norm 1 such that

$$\inf_{y\in\mathscr{M}_1} \left\|x_2 - y\right\| \geq 1.$$

We denote by $\mathscr{M}_2 = \text{span}\{x_1, x_2\}$ and continue by induction. Assume that we found $x_1, \dots, x_n$ of norm 1 such that

$$\inf_{y\in\mathscr{M}_k} \left\|x_{k+1} - y\right\| \geq 1$$

for all $0 \leq k \leq n-1$, where $\mathscr{M}_k = \text{span}\{x_1, \dots, x_k\}$ when $k \geq 1$ and $M_0 = \{0\}$. Using F. Riesz's theorem again we can find x_{n+1} of norm 1 such that

$$\inf_{y\in\mathscr{M}_n} \left\|x_{n+1} - y\right\| \geq 1.$$

and we define $\mathscr{M}_{n+1} = \text{span}\{x_1, \dots, x_{n+1}\}$.
If $\mathscr{M}_n = \mathscr{X}$ for some n then $\mathscr{X}$ is finite dimensional which is impossible. Hence, we have constructed a sequence of vectors of norm 1, $\{x_n\}_{n\in\mathbb{N}}$, and a sequence of finite dimensional spaces, $\{\mathscr{M}_n\}_{n\in\mathbb{N}}$ such that

$$\inf_{y\in\mathscr{M}_n} \left\|x_{n+1} - y\right\| \geq 1, \qquad \text{and} \qquad \mathscr{M}_n \underset{\neq}{\subset} \mathscr{M}_{n+1}.$$

For any $n \neq m \in \mathbb{N}$ we see that

$$\|x_n - x_m\| \geq \inf_{y\in\mathscr{M}_{\min\{n,m\}}} \left\|x_{\max\{n,m\}} - y\right\| \geq \inf_{y\in\mathscr{M}_{\max\{n,m\}-1}} \left\|x_{\max\{n,m\}} - y\right\| \geq 1,$$

Proving the first statement of the exercise.
Defining the sequence $\left\{x_n^{(M)}\right\}_{n\in\mathbb{N}}$ for an $M > 0$ by $x_n^{(M)} = Mx_n$ gives us a sequence of vectors of norm M with the property

$$\left\|x_n^{(M)} - x_m^{(M)}\right\| \geq M.$$

The above implies that the sequence has no subsequence that is Cauchy, which in turn shows that the sequence can have no converging subsequence. Thus, $\overline{B}_M(0)$ can't be compact for any $M > 0$ and the exercise is now complete.

Solution to Question 4. It is worth to mention that since $f \in C[a,b]$, the Fundamental Theorem of Calculus assures us that $Tf \in C[a,b]$ (and more than that - Tf is differentiable on (a,b)). The linearity of T is an immediate consequence of the linearity of the integral so we won't show it here (though you do need to show it when asked such questions).
To show the boundedness we notice that for any $f \in C[a,b]$ and any $x \in [a,b]$

$$\left|Tf(x)\right| \leq \int_a^x \left|f(t)\right| dt \leq \left\|f\right\|_\infty (x-a)$$

Thus
$$\|Tf\|_\infty \le \sup_{x\in[a,b]} \|f\|_\infty (x-a) = (b-a)\|f\|_\infty .$$
Thus, $\|T\| \le (b-a)$ which shows the desired boundedness.