Problem Class 4

Exercise 1. Prove the following statement: Let $\mathscr{X}$ be a Banach space with Schauder basis $\mathscr{B} = \{e_n\}_{n\in\mathbb{N}}$ and denote by

$$\mathscr{X}_n = \overline{\operatorname{span}\{e_k\}_{k\in\mathbb{N},\, k\neq n}}.$$

If for every $n \in \mathbb{N}$ we have that $e_n \notin \mathscr{X}_n$ then there exists a unique sequence $\left\{f^{(n)}\right\}_{n\in\mathbb{N}} \subset \mathscr{X}^*$ such that $f^{(n)}\left(e_j\right) = \delta_{n,j}$. Moreover, denoting by

$$d_n = \inf_{y\in\mathscr{X}_n} \left\|e_n - y\right\|$$

we have that $d_n > 0$ for every $n \in \mathbb{N}$ and $\left\|f^{(n)}\right\| = \frac{1}{d_n}$.

Exercise 2. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces. A linear operator T is called *compact* if for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathscr{X}$ there exists a subsequence $\left\{x_{n_k}\right\}_{k\in\mathbb{N}} \in \mathscr{X}$ such that the sequence $\left\{Tx_{n_k}\right\}_{k\in\mathbb{N}} \in \mathscr{Y}$ converges. Show that if T is compact then for any bounded set $M \subseteq X$ we have that $\overline{T(M)}$ is compact in $\mathscr{Y}$ and conclude that if T is compact then it is bounded.
Hint: You may use the following known fact from the theory of metric spaces: A set A in a metric space satisfies that $\overline{A}$ is compact[1] *if and only if any sequence of elements in A has a subsequence that converges in the space.*

Exercise 3. Show that there exists a functional in $\ell_\infty(\mathbb{N})^*$ that is not of the form $f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$.
Hint: Using the fact that

$$f_{\sum_{n=1}^N \alpha_n \boldsymbol{e}_n} = \sum_{n=1}^N \overline{\alpha_n} f_{\boldsymbol{e}_n}$$

conclude that if every $\ell_\infty(\mathbb{N})^*$ is of the form $f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$ then $\ell_\infty(\mathbb{N})^*$ is separable.

Exercise 4. Let $\mathscr{H}$ be a Hilbert space. Show that the following conditions are equivalent:

(i) $x_n \underset{n\to\infty}{\longrightarrow} x$.

(ii) $x_n \underset{n\to\infty}{\overset{w}{\longrightarrow}} x$ and $\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\|$.

[1] In this case we say that A is *pre-compact.*