Problem Class 4 Solution

Solution to Question 1. Since $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis, every $x \in \mathcal{X}$ can be written uniquely as

$$x = \sum_{n\in\mathbb{N}} \alpha_n(x) e_n$$

for some sequence of scalars $\{\alpha_n(x)\}_{n\in\mathbb{N}}$. We define

$$f^{(n)}(x) = \alpha_n(x),$$

and find that $f^{(n)}$ is linear due to the uniqueness of the expansion of the vector x with respect to the basis (try to show this!). We thus turn our attention to the positivity of d_n and the boundedness of $f^{(n)}$.
The fact that $d_n > 0$ follows directly from the assumption that $e_n \notin \mathcal{X}_n$. Indeed, if the infimum is zero we can find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{X}_n$ such that

$$\|e_n - x_k\| \leq \inf_{y\in\mathcal{X}_n} \|e_n - y\| + \frac{1}{k} = \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0.$$

This implies that e_n is a limit of a sequence of elements from $\mathcal{X}_n$, i.e. e_n is in $\overline{\mathcal{X}_n} = \mathcal{X}_n$, which is a contradiction.
Next we focus on the boundedness of $f^{(n)}$. We start by noticing that

$$\|x\| = \left\| \sum_{k\in\mathbb{N}} \alpha_k(x) e_n \right\| = \left\| \alpha_n(x) e_n + \sum_{k\neq n} \alpha_k(x) e_k \right\|.$$

Remembering that $f^{(n)}(x) = \alpha_n(x)$ and noticing that $\sum_{k\neq n} \alpha_k(x) e_k$ belongs to $\mathcal{X}_n$[1], we find that:
If $\alpha_n(x) \neq 0$ then

$$\|x\| = |\alpha_n(x)| \left\| e_n + \frac{\sum_{k\neq n} \alpha_k(x) e_k}{\alpha_n(x)} \right\| = \left|f^{(n)}(x)\right| \left\| e_n - \underbrace{\left(-\frac{\sum_{k\neq n} \alpha_k(x) e_k}{\alpha_n(x)} \right)}_{\in \mathcal{X}_n} \right\| \geq \left|f^{(n)}(x)\right| d_n.$$

Consequently, if $\alpha_n(x) \neq 0$ we find that $\left|f^{(n)}(x)\right| \leq \frac{\|x\|}{d_n}$.
If $\alpha_n(x) = 0$, on the other hand, then $f^{(n)}(x) = 0$ and the above remains true. We conclude that for all $x \in \mathcal{X}$

$$\left|f^{(n)}(x)\right| \leq \frac{\|x\|}{d_n},$$

[1] Since $\sum_{k\neq n} \alpha_k(x) e_k$ converges to $x - \alpha_n(x) e_n$ and $\sum_{k\neq n, k=1}^{N} \alpha_k(x) e_k \in \mathcal{X}_n$.

which shows that $f^{(n)} \in \mathcal{X}^*$ and $\left\|f^{(n)}\right\| \leq \frac{1}{d_n}$.
To show that this is indeed the norm of $f^{(n)}$ we find for any $\varepsilon > 0$ an element y_ε in $\mathcal{X}_n$ such that

$$d_n \leq \left\|e_n - y_\varepsilon\right\| \leq (1+\varepsilon)\, d_n$$

and notice that

$$\left|f^{(n)}(e_n - y_\varepsilon)\right| = \left|f^{(n)}(e_n)\right| = 1 = \frac{\left\|e_n - y_\varepsilon\right\|}{\left\|e_n - y_\varepsilon\right\|} \geq \frac{\left\|e_n - y_\varepsilon\right\|}{(1+\varepsilon)\, d_n},$$

and as such

$$\left\|f^{(n)}\right\| = \sup_{x\neq 0} \frac{\left|f^{(n)}(x)\right|}{\|x\|} \geq \frac{1}{(1+\varepsilon)\, d_n}.$$

As ε is arbitrary we conclude that $\left\|f^{(n)}\right\| \geq \frac{1}{d_n}$ and together with the upper bound on this norm we find that $\left\|f^{(n)}\right\| = \frac{1}{d_n}$.
Lastly, the uniqueness of the sequences $\left\{f^{(n)}\right\}_{n\in\mathbb{N}} \subset \mathcal{X}^*$ follows from the fact that if $g^{(n)} \subset \mathcal{X}^*$ satisfies $g^{(n)}(e_k) = \delta_{n,k}$ then due to its continuity

$$g^{(n)}(x) = g^{(n)}\left(\sum_{k\in\mathbb{N}} \alpha_k(x) e_k\right) = g^{(n)}\left(\lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k(x) e_k\right)$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \alpha_k(x) g^{(n)}(e_k) = \alpha_k(x) = f^{(n)}(x).$$

As the above holds for any $x \in \mathcal{X}$ we have that $g^{(n)} = f^{(n)}$. The proof is now complete.

Solution to Question 2. Using the given hint we see that in order to show our desired result, we only need to show that any sequence $\left\{y_n\right\}_{n\in\mathbb{N}} \in T(M)$ has a converging subsequence in $\mathcal{Y}$. Indeed, given $\left\{y_n\right\}_{n\in\mathbb{N}} \in T(M)$ we can find $\{x_n\}_{n\in\mathbb{N}} \in M$ such that $y_n = Tx_n$. As M is bounded and T is compact, we can find a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$, for which $y_{n_k} = Tx_{n_k}$ converges.
To show the continuity of T we notice that $M = \{x \in \mathcal{X} \mid \|x\| = 1\}$ is a bounded set. Since $T(M) \subseteq \overline{T(M)}$ and $\overline{T(M)}$ is compact we find that as compact sets in metric spaces are always bounded we must have that for any vector $x \in M$

$$\|Tx\| \leq C$$

for some $C > 0$. This implies that $\|T\| \leq C$ and as such the operator is bounded.

Solution to Question 3. We have seen in class that if $f \in \ell_\infty(\mathbb{N})^*$ is of the form $f = f_{\boldsymbol{b}}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{N})$ then

$$\left\|f_{\boldsymbol{b}}\right\| \leq \|\boldsymbol{b}\|_1. \tag{1}$$

Let $\mathscr{B} = \{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ be the standard Scahuder basis of $\ell_1(\mathbb{N})$ and denote by $M = \left\{f_{\boldsymbol{e}_n}\right\}_{n\in\mathbb{N}}$. We claim that

$$f_{\sum_{n=1}^{N}\alpha_n\boldsymbol{e}_n} = \sum_{n=1}^{N}\overline{\alpha_n}f_{\boldsymbol{e}_n}$$

for any scalars $\alpha_1,\dots,\alpha_N$. Indeed, given any $\boldsymbol{a}\in\ell_\infty(\mathbb{N})$ we see that

$$f_{\sum_{n=1}^{N}\alpha_n\boldsymbol{e}_n}(\boldsymbol{a}) = \sum_{j\in\mathbb{N}}a_j\overline{\left(\sum_{n=1}^{N}\alpha_n\boldsymbol{e}_n\right)_j} = \sum_{j\in\mathbb{N}}a_j\overline{\sum_{n=1}^{N}\alpha_n\delta_{n,j}}$$

$$= \sum_{j=1}^{N}a_j\overline{\alpha_j} = \sum_{n=1}^{N}\overline{\alpha_j}f_{e_n}(\boldsymbol{a}).$$

Since $\boldsymbol{a}\in\ell_\infty(\mathbb{N})$ was arbitrary we conclude the desired identity.
This, together with (1) shows that for any $\boldsymbol{b}\in\ell_1(\mathbb{N})$ we have that

$$\left\|f_{\boldsymbol{b}} - \sum_{n=1}^{N}\overline{b_n}f_{\boldsymbol{e}_n}\right\| = \left\|f_{\boldsymbol{b}} - f_{\sum_{n=1}^{N}b_n\boldsymbol{e}_n}\right\| \leq \left\|\boldsymbol{b} - \sum_{n=1}^{N}b_n\boldsymbol{e}_n\right\|_1 \underset{n\to\infty}{\longrightarrow} 0.$$

Consequently, if $\left\{f_{\boldsymbol{b}}\right\}_{\boldsymbol{b}\in\ell_1(\mathbb{N})} = \ell_\infty(\mathbb{N})^*$ we find that $\mathrm{span}M$ is dense in $\ell_\infty(\mathbb{N})^*$. Since M is countable we conclude that $\ell_\infty(\mathbb{N})^*$ is separable. From class we know that $\mathscr{X}^*$ is separable implies that $\mathscr{X}$ is also separable and as we know that $\ell_\infty(\mathbb{N})$ is not separable we have reached a contradiction.

Solution to Question 4. If (i) holds the by the continuity of norm

$$\|x_n\| \underset{n\to\infty}{\longrightarrow} \|x\|.$$

Moreover, for any $f\in\mathscr{H}^*$

$$0 \leq \left|f(x_n) - f(x)\right| = \left|f(x_n - x)\right| \leq \left\|f\right\|\|x_n - x\|,$$

which, using the squeezing lemma, shows that $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x)$ for any $f\in\mathscr{H}^*$, i.e.

$$x_n \underset{n\to\infty}{\overset{w}{\longrightarrow}} x.$$

Note that the above proof holds in any normed space and not only in a Hilbert space.

Assume now that (ii) holds. We have that

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\mathrm{Re}\langle x_n, x\rangle + \|x\|^2.$$

Since the norms converge we find that

$$\|x_n\|^2 \underset{n\to\infty}{\longrightarrow} \|x\|^2.$$

Since the sequence converges weakly we have that

$$2\mathrm{Re}\,\langle x_n, x\rangle \underset{n\to\infty}{\longrightarrow} 2\mathrm{Re}\,\langle x, x\rangle = 2\,\|x\|^2 .$$

We conclude that

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\mathrm{Re}\,\langle x_n, x\rangle + \|x\|^2 \underset{n\to\infty}{\longrightarrow} \|x\|^2 - 2\,\|x\|^2 + \|x\|^2 = 0,$$

showing the desired result.