#Functional

Revision Class

Exercise 1. (a) State the Hahn-Banach theorem.

(b) Let $\mathcal{X}$ be a Banach space over a field $\mathbb{F}$. For a given $0 \neq x \in \mathcal{X}$ define

$$\widetilde{f}_x : \operatorname{span}\{x\} \to \mathbb{F}$$

by $\widetilde{f}_x(\alpha x) = \alpha \|x\|$ where $\alpha \in \mathbb{F}$. Show that $\widetilde{f}_x$ is a linear functional and that $\|\widetilde{f}_x\|_{\operatorname{span}\{x\}^*} = 1$.

(c) Prove that for any $0 \neq x \in \mathcal{X}$ there exists $f_x \in \mathcal{X}^*$ such that $f_x(x) = \|x\|$ and $\|f_x\|_{\mathcal{X}^*} = 1$.

Solution:

(a) Let X be a normed space and let Y be a subspace of X. Assume that g is a bounded linear functional on X then g can be extended to an element $\tilde{g} \in X^*$ s.t $\|\tilde{g}\|_{X^*} = \|g\|_{Y^*}$.

(b) Any element $y \in \operatorname{span}\{x\}$ can be uniquely written as $y = \alpha x$ for some $\alpha \in \mathbb{F} \Rightarrow \tilde{f}_x$ is well defined. Moreover, let $y_1, y_2 \in \operatorname{span}\{x\}$ then $\exists!\, \alpha_1, \alpha_2 \in \mathbb{F}$ s.t $y_1 = \alpha_1 x$ $y_2 = \alpha_2 x$ and

$$\tilde{f}_x(y_1 + y_2) = \tilde{f}_x(\alpha_1 x + \alpha_2 x) = \tilde{f}_x((\alpha_1 + \alpha_2) x)$$

$$(\alpha_1+\alpha_2)\|x\| = \alpha_1\|x\| + \alpha_2\|x\| = \tilde{f}_x(y_1) + \tilde{f}_x(y_2)$$

Similarly, $\forall \beta \in \mathbb{F}$ and $y = \alpha x \in \operatorname{span}\{x\}$

$$\tilde{f}_x(\beta y) = \tilde{f}_x(\beta(\alpha x)) = \tilde{f}_x((\beta\alpha)\, x)$$

$$= \beta\alpha\|x\| = \beta(\alpha\|x\|) = \beta \tilde{f}_x(y)$$

$\Rightarrow \tilde{f}_x$ is a linear functional.

$$|\tilde{f}_x(y)| \underset{\substack{y=\alpha x \\ \text{for some} \\ \alpha\in\mathbb{F}}}{=} |\tilde{f}_x(\alpha x)| = |\alpha\|x\||$$

$$= |\alpha|\,\|x\| = \|\alpha x\| = \|y\|$$

$\Rightarrow \tilde{f}_x$ is bounded. Moreover

$$\|\tilde{f}_x\|_{\operatorname{span}\{x\}^*} = \sup_{y\neq 0} \frac{|\tilde{f}_x(y)|}{\|y\|} = \sup_{y\neq 0} 1 = 1.$$

(c) For any $0\neq x\in X$ we use Hahn-Banach on

$\tilde{f}_x : \operatorname{span}\{x\} \longrightarrow \mathbb{F}$ and find an extension

$f_x : X \longrightarrow \mathbb{F}$ s.t. $\|f_x\|_{X^*} = \|\tilde{f}_x\|_{\operatorname{span}\{x\}^*} = 1$

and

$$f_x(x) \overset{x\in\operatorname{span}\{x\}}{=} \tilde{f}_x(x) = \|x\|.$$

Exercise 2. Consider the operator $T : \ell_\infty \to \ell_p$, with $1 \le p < \infty$, defined by

$$T(\boldsymbol{a}) = \left(a_1, \frac{a_2}{2^\alpha}, \dots, \frac{a_n}{n^\alpha}, \dots\right).$$

(a) Show that T is well defined when $\alpha > 1/p$. In that case also show that it is a linear operator and that it is bounded. Is T well defined when $\alpha = 1/p$?

(b) Show that for *any* $\alpha > 1/p$ the operator T is injective but not surjective.

(a) $T(a)$ to be defined in $\ell_p(\mathbb{N})$ we must have $\sum_{n\in\mathbb{N}} \left|\frac{a_n}{n^\alpha}\right|^p < \infty$

If $a \in \ell_\infty(\mathbb{N})$

$$\sum_{n\in\mathbb{N}} \frac{|a_n|^p}{n^{\alpha p}} \leq \|a\|_\infty^p \sum_{n\in\mathbb{N}} 1/n^{\alpha p} < \infty$$

when $\alpha p > 1$ or $\alpha > 1/p$.

$\rightarrow T$ is indeed well defined.

Once we'll show that T is linear the above also shows

$$\|T(a)\|_p = \left(\sum_{n\in\mathbb{N}} \frac{|a_n|^p}{n^{\alpha p}}\right)^{1/p} \leq \left(\sum_{n\in\mathbb{N}} 1/n^{\alpha p}\right)^{1/p} \|a\|_\infty$$

i.e T is bounded.

Next we'll show the linearity of T

$$T(a+b) = \left(a_1+b_1, \dots, \frac{a_n+b_n}{n^\alpha}, \dots\right)$$

$$\underset{\uparrow}{=} \left(a_1, \dots, \frac{a_n}{n^\alpha}, \dots\right) + \left(b_1, \dots, \frac{b_n}{n^\alpha}, \dots\right)$$

since we know that $T(a), T(b) \in \ell_p$

$$= T(a) + T(b).$$

Similarly, if $\beta \in \mathbb{F}$

same reason

$$T(\beta a) = \left(\beta a_1, \ldots \frac{\beta a_n}{n}, \right) = \beta\left(a_1, \ldots, \frac{a_n}{n^\alpha}, \ldots\right)$$

$$= \beta T(a).$$

T is indeed a bounded linear operator when $\alpha > 1/p$.

T is not well defined when $\alpha = 1/p$.

Indeed $a_0 = (1, 1, \ldots, 1, \ldots) \in \ell_\infty(\mathbb{N})$ but when $\alpha = 1/p$

$$T(a_0) = \left(1, \ldots, 1/n^{1/p}, \ldots\right) \notin \ell_p(\mathbb{N}).$$

(b) If $T(a) = T(b)$ then (by comparing components) $\frac{a_n}{n^\alpha} = \frac{b_n}{n^\alpha} \quad \forall n \in \mathbb{N}$

$\Rightarrow a_n = b_n \ \forall n \in \mathbb{N}$, i.e. $a = b$. T is injective.

To show lack of surjectivity we choose $1/p < \beta < \alpha$, then

$$b = \left(1, \ldots, 1/n^\beta, \ldots\right) \in \ell_p$$

If $\exists a \in \ell_\infty(\mathbb{N})$ s.t. $T(a) = b$ then

$$\frac{a_n}{n^\alpha} = 1/n^\beta \Rightarrow a_n = n^{\alpha - \beta} \xrightarrow[n \to \infty]{} \infty$$

impossible!

Exercise 5. Consider the subset $\mathcal{H} \subset \ell_2$ given by

$$\mathcal{H} = \left\{ \boldsymbol{a} \in \ell_2 \mid \sum_{n\in\mathbb{N}} n^2 |a_n|^2 < \infty \right\}.$$

(a) Is $\mathcal{H}$ closed with respect to the norm of ℓ_2? Prove your claim.

(b) Let B be the set

$$B = \left\{ \boldsymbol{a} \in \mathcal{H} \mid \sum_{n\in\mathbb{N}} n^2 |a_n|^2 \le 1 \right\} \subset \mathcal{H}.$$

Show that for any $\boldsymbol{a} \in B$ we have that

$$\sum_{n\ge N} |a_n|^2 \le \frac{1}{N^2}$$

and then prove that if $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}}$ is a sequence in B such that

$$(a_n)_j \underset{n\to\infty}{\longrightarrow} a_j, \qquad \forall j \in \mathbb{N}$$

for some $\boldsymbol{a} \in B$ (component-wise convergence) then

$$\|\boldsymbol{a}_n - \boldsymbol{a}\|_{\ell_2} \underset{n\to\infty}{\longrightarrow} 0.$$

Solution:

(a) The seqence

$$a_n = (1, \dots \tfrac{1}{n}, 0, \dots, 0) \in \ell_2(\mathbb{N})$$

and $a_n \xrightarrow{\ell_2} a$ $\quad a = (1, \tfrac{1}{2}, \dots, \tfrac{1}{n}, \dots\dots) = \{\tfrac{1}{n}\}_{n\in\mathbb{N}}$

$a_n \in \mathcal{H}$ since

$$\sum_{j=1}^{\infty} j^2 (a_n)_j^2 = \sum_{j=1}^{n} 1 = n$$

but $a \notin \mathcal{H}$ $\left(\sum_j j^2 a_j^2 = \sum_{j=1}^{\infty} 1 = \infty \right)$

(b)

$$\sum_{n\geq N}' |a_n|^2 = \sum_{n \geq N} \frac{n^2 |a_n|^2}{n^2} \leq \frac{1}{N^2} \underbrace{\sum_{n\in\mathbb{N}}' n^2 |a_n|^2}_{a\in B} \leq 1$$

$$\leq 1/N^2.$$

If $(a_n)_j \xrightarrow{n\to\infty} a_j \quad \forall j$

$$\sum_{j=1}^{\infty}' |(a_n)_j - a_j|^2 = \underbrace{\sum_{j=1}^{N}' |(a_n)_j - a_j|^2}_{\downarrow\, n\to\infty \atop 0}$$

$$+ \underbrace{\sum_{j=N+1}^{\infty} |(a_n)_j - a_j|^2}$$

$$\leq 2\left(\sum_{N+1}^{\infty}' |a_{n,j}|^2 + \sum_{N+1}^{\infty} |a_j|^2\right)$$

$$\leq 4/(N+1)^2$$

Exercise 6. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces and let E be a given subset of $\mathscr{X}$. Let $T : \mathscr{X} \to \mathscr{Y}$ be a given bounded linear operator.

(a) Show that if $T|_E = 0$ then $T|_{\mathscr{M}} = 0$ where $\mathscr{M} = \overline{\operatorname{span} E}$.

(b) Let $\{E_n\}_{n\in\mathbb{N}}$ be a given sequence of subsets of $\mathscr{X}$. Show that if $\{T_n\}_{n\in\mathbb{N}} \in B(\mathscr{X}, \mathscr{Y})$ satisfy

$$T_n|_{E_j} = 0, \qquad \forall n \geq j,$$

then if $\{T_n\}_{n\in\mathbb{N}}$ converges to $T \in B(\mathscr{X}, \mathscr{Y})$ in the operator norm we have that

$$T|_{\overline{\operatorname{span} \cup_{n\in\mathbb{N}} E_n}} = 0.$$

See written solution