Functional Analysis and Applications IV-Revision Class

Exercise 1. (a) State the Hahn-Banach theorem.

(b) Let $\mathcal{X}$ be a Banach space over a field $\mathbb{F}$. For a given $0 \neq x \in \mathcal{X}$ define

$$\widetilde{f}_x : \operatorname{span}\{x\} \to \mathbb{F}$$

by $\widetilde{f}_x(\alpha x) = \alpha \|x\|$ where $\alpha \in \mathbb{F}$. Show that $\widetilde{f}_x$ is a linear functional and that $\|\widetilde{f}_x\|_{\operatorname{span}\{x\}^*} = 1$.

(c) Prove that for any $0 \neq x \in \mathcal{X}$ there exists $f_x \in \mathcal{X}^*$ such that $f_x(x) = \|x\|$ and $\|f_x\|_{\mathcal{X}^*} = 1$.

Solution. (a) Let $\mathcal{X}$ be a normed space and let $\mathcal{Y}$ be a subspace of $\mathcal{X}$. Assume that g is a bounded linear functional on $\mathcal{Y}$. Then there exists a bounded linear extension of g, $\widetilde{g}$, to all of $\mathcal{X}$ such that $\|g\|_{\mathcal{Y}^*} = \|\widetilde{g}\|_{\mathcal{X}^*}$, where

$$\|g\|_{\mathcal{Y}^*} = \begin{cases} \sup_{y \in \mathcal{Y},\, y \neq 0} \frac{|g(y)|}{\|y\|} & \mathcal{Y} \neq \{0\} \\ 0 & \mathcal{Y} = \{0\} \end{cases}.$$

(b) Since $\{x\}$ is a basis for $\operatorname{span}\{x\}$ we know that each vector in $\operatorname{span}\{x\}$ can be written uniquely as αx for some $\alpha \in \mathbb{F}$. Thus, $\widetilde{f}_x$ is well defined. Moreover

$$\widetilde{f}_x(\alpha_1 x + \alpha_2 x) = (\alpha_1 + \alpha_2)\|x\| = \alpha_1 \|x\| + \alpha_2 \|x\| = \widetilde{f}_x(\alpha_1 x) + \widetilde{f}_x(\alpha_2 x)$$

and for any $\beta \in \mathbb{F}$

$$\widetilde{f}_x\left(\beta(\alpha x)\right) = \widetilde{f}_x\left(\left(\beta\alpha\right)x\right) = \left(\beta\alpha\right)\|x\| = \beta(\alpha\|x\|) = \beta \widetilde{f}_x(\alpha x)$$

which shows that $\widetilde{f}_x$ is indeed a linear functional.

$$\|\widetilde{f}_x\|_{\operatorname{span}\{x\}^*} = \sup_{y \in \operatorname{span}\{x\} \setminus \{0\}} \frac{\left|\widetilde{f}_x(y)\right|}{\left\|y\right\|} = \sup_{\alpha \in \mathbb{F}\setminus\{0\}} \frac{\left|\widetilde{f}_x(\alpha x)\right|}{\|\alpha x\|}$$

$$= \sup_{\alpha \in \mathbb{F}\setminus\{0\}} \frac{|\alpha \|x\||}{|\alpha|\,\|x\|} = 1.$$

(c) According to the Hahn-Banach theorem, there exists $f_x \in \mathcal{X}^*$ such that $f_x|_{\operatorname{span}\{x\}} = \widetilde{f}_x$ and

$$f_x(x) = \widetilde{f}_x(x) = \|x\|.$$

Moreover, the operator norm of f_x equals that of $\widetilde{f}_x$, which shows the second part of the question.

□

Exercise 2. Consider the operator $T : \ell_\infty \to \ell_p$, with $1 \le p < \infty$, defined by

$$T(\boldsymbol{a}) = \left(a_1, \frac{a_2}{2^\alpha}, \dots, \frac{a_n}{n^\alpha}, \dots\right).$$

(a) Show that T is well defined when $\alpha > 1/p$. In that case also show that it is a linear operator and that it is bounded. Is T well defined when $\alpha = 1/p$?

(b) Show that for *any* $\alpha > 1/p$ the operator T is injective but not surjective.

Solution. (a) For any $\boldsymbol{a} \in \ell_\infty$ we have that

$$\sum_{n\in\mathbb{N}} \left|\frac{a_n}{n^\alpha}\right|^p = \sum_{n\in\mathbb{N}} \frac{|a_n|^p}{n^{\alpha p}} \le \|\boldsymbol{a}\|_\infty^p \sum_{n\in\mathbb{N}} \frac{1}{n^{\alpha p}} < \infty$$

since $\alpha p > 1$. This shows that T is well defined and, once we'll show that it is linear, it also shows that

$$\|T\boldsymbol{a}\|_p \le \left(\sum_{n\in\mathbb{N}} \frac{1}{n^{\alpha p}}\right)^{\frac{1}{p}} \|\boldsymbol{a}\|_\infty$$

which shows that T is bounded with $\|T\| \le \left(\sum_{n\in\mathbb{N}} \frac{1}{n^{\alpha p}}\right)^{\frac{1}{p}}$.

Given $\boldsymbol{a}, \boldsymbol{b} \in \ell_\infty$ we have that since $T(\boldsymbol{a})$ and $T(\boldsymbol{b})$ are in ℓ_p then

$$T(\boldsymbol{a} + \boldsymbol{b}) = \left(a_1 + b_1, \frac{a_2 + b_2}{2^\alpha}, \dots, \frac{a_n + b_n}{n^\alpha}, \dots\right)$$

$$= \left(a_1, \frac{a_2}{2^\alpha}, \dots, \frac{a_n}{n^\alpha}, \dots\right) + \left(b_1, \frac{b_2}{2^\alpha}, \dots, \frac{b_n}{n^\alpha}, \dots\right) = T\boldsymbol{a} + T\boldsymbol{b}$$

and for any $\alpha \in \mathbb{F}$

$$T(\alpha \boldsymbol{a}) = \left(\alpha a_1, \frac{\alpha a_2}{2^\alpha}, \dots, \frac{\alpha a_n}{n^\alpha}, \dots\right) = \alpha T \boldsymbol{a},$$

showing the linearity of T.

In the case $\alpha = \frac{1}{p}$ the operator is not defined on ℓ_∞. Indeed, $\boldsymbol{a} = (1, 1, 1, \dots,) \in \ell_\infty$ but

$$T(\boldsymbol{a}) = \left(1, \frac{1}{2^{\frac{1}{p}}}, \dots, \frac{1}{n^{\frac{1}{p}}}, \dots\right) \notin \ell_p.$$

(b) We have that

$$T(\boldsymbol{a}) = T(\boldsymbol{b})$$

if and only if $\frac{a_n}{n^\alpha} = \frac{b_n}{n^\alpha}$ for all $n \in \mathbb{N}$ (pointwise equality in the sequence space), or equivalently $a_n = b_n$ for all $n \in \mathbb{N}$. This implies that $\boldsymbol{a} = \boldsymbol{b}$ which shows the injectivity.

To show that the map is not surjective we fix $\frac{1}{p} < \beta < \alpha$. As was shown above the vector

$$\boldsymbol{x} = \left(1, \frac{1}{2^\beta}, \dots, \frac{1}{n^\beta}, \dots\right)$$

belongs to ℓ_p however if $T(\boldsymbol{a}) = \boldsymbol{x}$ then $\frac{a_n}{n^\alpha} = \frac{1}{n^\beta}$ for all $n \in \mathbb{N}$ or equivalently $a_n = n^{\alpha-\beta}$. Since $\alpha > \beta$ the vector $\boldsymbol{a}$ can't be in ℓ_∞, which shows the desired result.

□

Exercise 3. Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces.

(a) Define what it means for a linear operator $T : \mathscr{X} \to \mathscr{Y}$ to be compact.

(b) Let $f : \mathscr{X} \to \mathbb{C}$ be an unbounded linear functional (where we assume that such a functional exists). For a fixed $0 \neq x_0 \in \mathscr{X}$ let $T : \mathscr{X} \to \mathscr{X}$ be defined by $Tx = f(x)x_0$. Show that T has finite rank (i.e. $\dim(\mathscr{R}(T)) < \infty$) but is not compact.

Solution. (a) T is compact if it maps bounded sets to pre-compact sets. Equivalently, T is compact if for every bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ the sequence $\{Tx_n\}_{n\in\mathbb{N}} \subset \mathscr{Y}$ has a convergent subsequence.

(b) Since $\mathscr{R}(T) = \operatorname{span}\{x_0\}$ has dimension $1 < \infty$, T has finite rank. Since f is unbounded, we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ with $\|x_n\| = 1$ and $|f(x_n)| \to \infty$ as $n \to \infty$. If T were compact, $\{Tx_n\}_{n\in\mathbb{N}} \subset \mathscr{X}$ would have a convergent subsequence, but since $\|Tx_n\| = \|f(x_n)x_0\| = |f(x_n)|\|x_0\| \to \infty$, this cannot be true. Thus T is not compact.

□

Exercise 4. Let $u : \mathbb{R} \to \mathbb{R}$ be defined by $u(x) = (1 + |x|)^{-1}$.

(a) For which $1 \leq p \leq \infty$ is $u \in W^{1,p}(\mathbb{R})$? (note the change here from $W_0^{1,p}$ in the original exam).

(b) Give an explicit $f \in H^{-1}(\mathbb{R})$ such that $f(u) = 1$.

Solution. (a) The function u is continuous on $\mathbb{R}$ and differentiable at all $x \neq 0$, with

$$u'(x) = -\operatorname{sgn}(x)(1 + |x|)^{-2}.$$

From this we see that u is Lipschitz and hence from the lectures we know that it is weakly differentiable.

Thus $u \in W^{1,p}(\mathbb{R})$ if and only if $u, u' \in L^p(\mathbb{R})$. Since u is even, this is equivalent to proving that $u, u' \in L^p([0,\infty))$. For $p = \infty$ this is obviously true. Now let $1 \leq p < \infty$. We have

$$\int_0^\infty |u|^p \, \mathrm{d}x = \int_0^\infty (1+x)^{-p} \, \mathrm{d}x = \begin{cases} -\frac{1}{p-1}(1+x)^{-p+1}|_{x=0}^{x=\infty} = \frac{1}{p-1}, & p > 1, \\ \log(1+x)|_{x=0}^{x=\infty} = \infty, & p = 1. \end{cases}$$

Thus $u \in L^p(\mathbb{R})$ for $p > 1$ but not for $p = 1$. Note that since $|u'| = |u|^2$, we have $u' \in L^p(\mathbb{R})$ if and only if $u \in L^{2p}(\mathbb{R})$, which is true for all $p > 1/2$. Thus $u \in W^{1,p}(\mathbb{R})$ for all $1 < p \leq \infty$ but not for $p = 1$.

(b) $H^{-1}(\mathbb{R})$ is the dual space of $H^1(\mathbb{R})$. We can take any $g \in H^{-1}(\mathbb{R})$ with $g(u) \neq 0$ and then define

$$f(v) := \frac{g(v)}{g(u)}, \quad v \in H^1(\mathbb{R}).$$

We know that (from Riesz's Representation Theorem) any $g \in H^{-1}(\mathbb{R})$ is of the form $g(v) = \langle v, h \rangle_{H^1}$ for fixed $h \in H^1(\mathbb{R})$. We can take for example $h = u$, since then $g(u) = \|u\|_{H^1}^2 \geq \|u\|_{L^2}^2 > 0$. Thus

$$f(v) = \frac{\langle v, u \rangle_{H^1}}{\|u\|_{H^1}^2}.$$

□

Exercise 5. Consider the subset $\mathcal{H} \subset \ell_2$ given by

$$\mathcal{H} = \left\{ \boldsymbol{a} \in \ell_2 \mid \sum_{n \in \mathbb{N}} n^2 |a_n|^2 < \infty \right\}.$$

(a) Is $\mathcal{H}$ closed with respect to the norm of ℓ_2? Prove your claim.

(b) Let B be the set

$$B = \left\{ \boldsymbol{a} \in \mathcal{H} \mid \sum_{n \in \mathbb{N}} n^2 |a_n|^2 \leq 1 \right\} \subset \mathcal{H}.$$

Show that for any $\boldsymbol{a} \in B$ we have that

$$\sum_{n \geq N} |a_n|^2 \leq \frac{1}{N^2}$$

and then prove that if $\{\boldsymbol{a}_n\}_{n \in \mathbb{N}}$ is a sequence in B such that

$$(\boldsymbol{a}_n)_j \underset{n \to \infty}{\longrightarrow} a_j, \qquad \forall j \in \mathbb{N}$$

for some $\boldsymbol{a} \in B$ (component-wise convergence) then

$$\|\boldsymbol{a}_n - \boldsymbol{a}\|_{\ell_2} \underset{n \to \infty}{\longrightarrow} 0.$$

Solution. (a) $\mathcal{H}$ is not closed. Indeed, the sequence $\{\boldsymbol{a}_n\}_{n \in \mathbb{N}}$ defined by

$$\boldsymbol{a}_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$$

belongs to $\mathcal{H}$ since

$$\sum_{j \in \mathbb{N}} j^2 \left|(\boldsymbol{a}_n)_j\right|^2 = n$$

and it converges to $\boldsymbol{a} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ (this follows from the fact that $\{\boldsymbol{e}_n\}_{n\in\mathbb{N}}$ is a Schauder basis for ℓ_2). This sequence is not in $\mathscr{H}$ since

$$\sum_{n\in\mathbb{N}} n^2 |a_n|^2 = \sum_{n\in\mathbb{N}} 1 = \infty.$$

(b) For any $\boldsymbol{a} \in B$ we find that

$$\sum_{n\geq N} |a_n|^2 \leq \sum_{n\geq N} \frac{n^2}{N^2} |a_n|^2 \leq \frac{\sum_{n\in\mathbb{N}} n^2 |a_n|^2}{N^2} \leq \frac{1}{N^2}.$$

Assume now that $\{\boldsymbol{a}_n\}_{n\in\mathbb{N}} \in B$ converges component wise to $\boldsymbol{a} \in B$. Then for any $N \in \mathbb{N}$

$$\begin{aligned} \|\boldsymbol{a}_n - \boldsymbol{a}\|^2 &= \sum_{j=1}^{N} \left|(a_n)_j - a_j\right|^2 + \sum_{j=N+1}^{\infty} \left|(a_n)_j - a_j\right|^2 \\ &\leq \sum_{j=1}^{N} \left|(a_n)_j - a_j\right|^2 + 2\sum_{j=N+1}^{\infty} \left(\left|(a_n)_j\right|^2 + \left|a_j\right|^2\right) \\ &\leq \sum_{j=1}^{N} \left|(a_n)_j - a_j\right|^2 + \frac{4}{(N+1)^2}. \end{aligned}$$

Thus, due to the component convergence, we find that for any $N \in \mathbb{N}$

$$\limsup_{n\to\infty} \|\boldsymbol{a}_n - \boldsymbol{a}\|^2 \leq \limsup_{n\to\infty} \sum_{j=1}^{N} \left|(a_n)_j - a_j\right|^2 + \frac{4}{(N+1)^2} = \frac{4}{(N+1)^2}.$$

As $N \in \mathbb{N}$ is arbitrary we can take it to infinity and conclude that

$$\lim_{n\to\infty} \|\boldsymbol{a}_n - \boldsymbol{a}\|^2 = 0.$$

□

Exercise 6. Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces and let E be a given subset of $\mathscr{X}$. Let $T : \mathscr{X} \to \mathscr{Y}$ be a given bounded linear operator.

(a) Show that if $T|_E = 0$ then $T|_{\mathscr{M}} = 0$ where $\mathscr{M} = \overline{\text{span}E}$.

(b) Let $\{E_n\}_{n\in\mathbb{N}}$ be a given sequence of subsets of $\mathscr{X}$. Show that if $\{T_n\}_{n\in\mathbb{N}} \in B\left(\mathscr{X},\mathscr{Y}\right)$ satisfy

$$T_n|_{E_j} = 0, \qquad \forall n \geq j,$$

then if $\{T_n\}_{n\in\mathbb{N}}$ converges to $T \in B\left(\mathscr{X},\mathscr{Y}\right)$ in the operator norm we have that

$$T|_{\overline{\text{span}\cup_{n\in\mathbb{N}} E_n}} = 0.$$

Solution. (a) Let $x \in \text{span}E$. Then there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$ and scalars $\alpha_1, \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i.$$

Since $T|_E = 0$ and T is a linear operator we see that

$$Tx = \sum_{i=1}^{n} \alpha_i T x_i = \sum_{i=1}^{n} \alpha_i 0 = 0.$$

Thus we find that $T|_{\text{span}E} = 0$. Next, given $x \in \mathscr{M}$ we know that we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ from span$E$ that converges to x.

Since T is bounded (and as such continuous) we have that

$$Tx = \lim_{n\to\infty} Tx_n \underset{T|_{\text{span}E}=0}{=} \lim_{n\to\infty} 0 = 0.$$

As x was arbitrary we conclude that $T|_{\mathscr{M}} = 0$.

(b) Using 6.1 we see that it is enough to show that $T|_{\cup_{n\in\mathbb{N}} E_n} = 0$. Indeed, let $x \in \cup_{n\in\mathbb{N}} E_n$. Then there exists $n_0 \in \mathbb{N}$ such that $x \in E_{n_0}$. Thus, for any $n \ge n_0$ we have that $T_n x = 0$.

Since T_n converges to T in the operator norm we see that for any $n \ge n_0$

$$\|Tx\| = \|Tx - T_n x\| \le \|T - T_n\| \, \|x\|$$

As the above holds for any such n we can take n to infinity and conclude that $\|Tx\| = 0$, i.e. $Tx = 0$. Since x was arbitrary we find that $T|_{\cup_{n\in\mathbb{N}} E_n} = 0$ which completes the proof.

□

Exercise 7. Consider $T : \ell_2 \to \ell_2$ defined by

$$T(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = (0, x_1, 0, x_3, 0, x_5, \dots).$$

(a) Compute T^2.

(b) Find $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

Solution. (a) Let $x \in \ell_2$. Then $T^2 x = T(0, x_1, 0, x_3, 0, x_5, \dots) = (0, 0, 0, \dots)$. Hence $T^2 = 0$ in ℓ_2.

(b) The point $\lambda = 0$ is an eigenvalue because for example $Te_2 = 0$.

Quick way: The spectral mapping theorem for polynomials implies $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$. Since $\sigma(T^2) = \sigma(0) = 0$, the spectrum $\sigma(T)$ contains only $\lambda \in \mathbb{C}$ such that $\lambda^2 = 0$, whose only solution is $\lambda = 0$. Thus $\sigma(T) = \{0\}$, $\rho(T) = \mathbb{C}\setminus\{0\}$.

Longer way to prove $\mathbb{C}\setminus\{0\} \subset \rho(T)$: If $\lambda \neq 0$, then $T - \lambda$ is injective because $(T-\lambda)x = 0$ means (consider separately odd and even indices n):

$$-\lambda x_{2j-1} = 0, \quad x_{2j-1} - \lambda x_{2j} = 0, \quad \text{for } j \in \mathbb{N}.$$

The first equation implies that $x_n = 0$ for all odd $n = 2j-1$, and inserting this into the second equation implies $x_n = 0$ for all even $n = 2j$. Thus $x = 0$.

To prove surjectivity, let $(T-\lambda)x = y$. This means

$$-\lambda x_{2j-1} = y_{2j-1}, \quad x_{2j-1} - \lambda x_{2j} = y_{2j}, \quad \text{for } j \in \mathbb{N}.$$

Thus $x_{2j-1} = -\lambda^{-1} y_{2j-1}$ and $x_{2j} = \lambda^{-1}(x_{2j-1} - y_{2j}) = -\lambda^{-2} y_{2j-1} - \lambda^{-1} y_{2j}$. This x belongs to ℓ_2 because

$$\sum_{n=1}^{\infty} |x_n|^2 = \sum_{j=1}^{\infty} \left(|x_{2j-1}|^2 + |x_{2j}|^2\right) = \sum_{j=1}^{\infty} \left(|-\lambda^{-1} y_{2j-1}|^2 + |-\lambda^{-2} y_{2j-1} - \lambda^{-1} y_{2j}|^2\right)$$
$$\le 2|\lambda|^{-2} \|y\|^2 + 2|\lambda|^{-4} \|y\|^2 < \infty,$$

where we have used $\|u+v\|^2 \le 2(\|u\|^2 + \|v\|^2)$.

Altogether,

$$\sigma_p(T) = \sigma(T) = \{0\}, \quad \rho(T) = \mathbb{C} \setminus \{0\}, \quad \sigma_c(T) = \sigma_r(T) = \emptyset.$$

□

Exercise 8. Let $T = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ in the Hilbert space $L^2([0,1])$ with

$$\mathcal{D}(T) = \left\{ f \in L^2([0,1]) : f \text{ absolutely continuous on } [0,1],\ f' \in L^2([0,1]),\ f(0) = \mathrm{i} f(1) \right\}.$$

(a) Show that T is densely defined.
(b) Show that T is symmetric.
(c) Show that T is selfadjoint. You may use without proof that the general solution of $f' - \mu f = g$ (for given $\mu \in \mathbb{C}$ and $g \in L^2([0,1])$) is

$$f(x) = \exp(\mu x)\left(C + \int_0^x \exp(-\mu t) g(t)\,\mathrm{d}t\right),$$

for a constant $C \in \mathbb{C}$.

Solution. (a) We know that $C_0^\infty((0,1))$ is dense in $L^2([0,1])$. Since $C_0^\infty((0,1)) \subset \mathcal{D}(T) \subset L^2([0,1])$, the closure of $\mathcal{D}(T)$ is also $L^2([0,1])$.

(b) From the lectures we know that it suffices to prove that $\langle Tf, f\rangle \in \mathbb{R}$ for all $f \in \mathcal{D}(T)$. Indeed, integration by parts yields

$$\langle Tf, f\rangle = \mathrm{i}\int_0^1 f'(x)\overline{f(x)}\,\mathrm{d}x = \mathrm{i}\frac{|f(x)|^2}{2}\Big|_{x=0}^{x=1} - \mathrm{i}\int_0^1 f(x)\overline{f'(x)}\,\mathrm{d}x$$
$$= \int_0^1 f(x)\overline{\mathrm{i}f'(x)}\,\mathrm{d}x = \overline{\langle Tf, f\rangle},$$

where we have used $|f(x)|^2|_{x=0}^{x=1} = |f(1)|^2 - |f(0)|^2 = 0$ by the assumption $f(0) = \mathrm{i}f(1)$. Since a complex number is equal to its complex conjugate if and only if it is real, we get $\langle Tf, f\rangle \in \mathbb{R}$.

(c) Once we know that an operator is symmetric (we know it from (b)), we know from the lectures that it suffices to prove $\mathcal{R}(T \pm i) = L^2([0,1])$. [Instead of $\pm$i, one can take two different points, one in the upper and one in the lower complex half-plane.]

Let $g \in L^2([0,1])$. The equation $(T-\lambda)f = \mathrm{i}f' - \lambda f = g$ has the general solution (using a variation of constants method – see hint)

$$f(x) = \mathrm{e}^{\frac{\lambda}{\mathrm{i}}x}\left(C + \int_0^x \mathrm{e}^{-\frac{\lambda}{\mathrm{i}}t} g(t)\,\mathrm{d}t\right).$$

Now we choose the constant C such that f satisfies the boundary condition $f(0) = \mathrm{i}f(1)$. This means

$$C = f(0) = \mathrm{i}f(1) = \mathrm{i}\mathrm{e}^{\frac{\lambda}{\mathrm{i}}}\left(C + \int_0^1 \mathrm{e}^{-\frac{\lambda}{\mathrm{i}}t} g(t)\,\mathrm{d}t\right),$$

hence, if $1 - \mathrm{i}\mathrm{e}^{\frac{\lambda}{\mathrm{i}}} \neq 0$,

$$C = \frac{\mathrm{i}\mathrm{e}^{\frac{\lambda}{\mathrm{i}}} \int_0^1 \mathrm{e}^{-\frac{\lambda}{\mathrm{i}}t} g(t)\,\mathrm{d}t}{1 - \mathrm{i}\mathrm{e}^{\frac{\lambda}{\mathrm{i}}}}.$$

Note that for $\lambda = \pm\mathrm{i}$ we have $1 - \mathrm{i}\mathrm{e}^{\frac{\lambda}{\mathrm{i}}} = 1 - \mathrm{i}\mathrm{e}^{\pm 1} \neq 0$, so C is well-defined. The corresponding f is in $\mathcal{D}(T)$, which implies $\mathcal{R}(T \pm i) = L^2([0,1])$.

□