Partial Differential Equations III/V

Gappy Notes

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The following gappy notes are based on the official lecture notes, who have evolved over the years by M. Iacobelli, D. Bourne, J.F. Blowey, P.W. Dondl, and A.R. Yeates.

The LaTeX code of the last image in this note is a modification of one found with the help of Copilot.

Contents

CHAPTER 4

Poisson's Equation

Let $\Omega \subseteq \mathbb{R}^n$ be open. Poisson's equation is the PDE

$$-\Delta u = f$$

where $u : \Omega \to \mathbb{R}$ is the unknown and $f : \Omega \to \mathbb{R}$ is given.

DEFINITION 4.1 (Linear, second-order, elliptic PDEs). Let $\Omega \subseteq \mathbb{R}^n$ be open and $a_{ij}, b_j, c : \Omega \to \mathbb{R}$ for $i, j \in \{1, \dots, n\}$. Let A be the matrix–valued function defined by $[A(\boldsymbol{x})]_{ij} = a_{ij}(\boldsymbol{x})$, and let $\boldsymbol{b}$ be the vector–valued function defined by $[\boldsymbol{b}(\boldsymbol{x})]_j = b_j(\boldsymbol{x})$. Define the linear, second-order differential operator L by

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} b_j u_{x_j} + cu = -A : D^2 u + \boldsymbol{b} \cdot \nabla u + cu,$$

for $u : \Omega \to \mathbb{R}$. We say that L is *elliptic* if A is symmetric and uniformly positive definite, which means that $a_{ij}(\boldsymbol{x}) = a_{ji}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ and that there exists a constant $\alpha > 0$ such that $\boldsymbol{y}^T A(\boldsymbol{x}) \boldsymbol{y} \geq \alpha |\boldsymbol{y}|^2$ for all $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$. PDEs of the form $Lu(\boldsymbol{x}) = f(\boldsymbol{x})$ are called *elliptic* PDEs.

For example, for Poisson's equation $L = -\Delta$, $A = I$, $\boldsymbol{b} = \boldsymbol{0}$, $c = 0$, and $\alpha = 1$.

4.1. Poisson's Equation in $[a, b]$

In one dimension $\Delta u = u''$ and Poisson's equation has the form

$$-u'' = f$$

In the case where we consider Dirichlet boundary conditions, i.e. $u(a) = u(b) = 0$ you have shown that the solution to the equation is given by

$$u(x) = \int_a^x \frac{(y-a)(b-x)}{b-a} f(y)\, dy + \int_x^b \frac{(x-a)(b-y)}{b-a} f(y)\, dy.$$

which can be written as

$$u(x) = \int_a^b G(x, y) f(y)\, dy$$

where

$$G(x,y) = \begin{cases} \dfrac{(y-a)(b-x)}{b-a} & \text{if } y \le x, \\ \dfrac{(x-a)(b-y)}{b-a} & \text{if } y \ge x. \end{cases}$$

G a called the *Green's function* and is an important tool in the study of linear PDEs.

Properties of the Green function for the Dirichlet problem on $[a,b]$:

- G is symmetric, i.e. $G(x,y) = G(y,x)$.
- G is continuous.
- We have that

  $$G_y(x,y) = \begin{cases} \dfrac{b-x}{b-a} & \text{if } y < x, \\ -\dfrac{x-a}{b-a} & \text{if } y > x. \end{cases}$$

  showing that G_y is discontinuous on $y = x$. The same holds for $G_x(x,y)$.
- Outside of the diagonal $y = x$ we have that

  $$G_{xx}(x,y) = G_{yy}(x,y) = 0.$$

REMARK 4.2 (Green's functions). Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with smooth boundary. It can be shown that if $u \in C^2(\overline{\Omega})$ satisfies $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, where f and g are continuous, then there exists a Green's function G such that

$$u(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} - \int_{\partial\Omega} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, g(\boldsymbol{y})\, dS(\boldsymbol{y}). \tag{4.1}$$

See Evans (2010, Section 2.2.4). As above, the Green's function is symmetric and satisfies Laplace's equation in $\boldsymbol{x}$ and $\boldsymbol{y}$ away from the diagonal $\boldsymbol{y} = \boldsymbol{x}$.

4.2. Poisson equation in $\mathbb{R}^n$, $n \geq 2$

DEFINITION 4.3 (Fundamental solution). Let $n \geq 2$. The *fundamental solution of Poisson's equation in* $\mathbb{R}^n$ is the map $\Phi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ defined by

$$\Phi(\boldsymbol{x}) = \begin{cases} -\dfrac{1}{2\pi} \log|\boldsymbol{x}| & \text{if } n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|\boldsymbol{x}|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

where

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

and where $\Gamma : (0,\infty) \to \mathbb{R}$ is the Gamma function, which is defined by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

REMARK 4.4 (Facts about Γ and α).

- Γ can be considered an extension of the factorial. Using integration by parts one can show that for any $s > 0$
$$\Gamma(s+1) = s\Gamma(s).$$
One can also show that $\Gamma(1) = 1$ and as such
$$\Gamma(n) = (n-1)!$$
for any $n \in \mathbb{N}$. One can also show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- It can be shown that $\alpha(n)$ is the volume of the unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^n$:
$$\alpha(n) = \int_{B_1(\mathbf{0})} 1\, d\boldsymbol{x}.$$

As a consequence one can show that for any $R > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$

$$|B_R(\boldsymbol{x})| = \alpha(n) R^n.$$

Moreover, the surface area of $B_R(\boldsymbol{x})$ is given by

$$|\partial B_R(\boldsymbol{x})| = n\alpha(n) R^{n-1}.$$

LEMMA 4.5 (Properties of the fundamental solution).

(i) $\Delta\Phi(\boldsymbol{x}) = 0$ *for* $\boldsymbol{x} \neq \mathbf{0}$.

(ii) $\Phi(\boldsymbol{x}) \to \infty$ *as* $\boldsymbol{x} \to \mathbf{0}$.

(iii) Φ *has an integrable singularity at the origin: For any* $R > 0$*,*

$$\int_{B_R(\mathbf{0})} |\Phi(\boldsymbol{x})|\, d\boldsymbol{x} < \infty.$$

(iv) $\nabla\Phi$ *also has an integrable singularity at the origin: For any* $R > 0$*,*

$$\int_{B_R(\mathbf{0})} |\nabla\Phi(\boldsymbol{x})|\, d\boldsymbol{x} < \infty.$$

PROOF.

□

DEFINITION 4.6 (The function spaces L^1_{loc} and C^k_c).

(i) We define the the space of *locally integrable* functions on $\mathbb{R}^n$ to be

$$L^1_{\text{loc}}(\mathbb{R}^n) := \left\{ \varphi : \mathbb{R}^n \to \mathbb{R} : \int_K |\varphi(\boldsymbol{x})| \, d\boldsymbol{x} < \infty \text{ for any compact set } K \subset \mathbb{R}^n \right\}.$$

(ii) Let k be a nonnegative integer. We let

$$C^k_c(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{R} : f \in C^k(\mathbb{R}^n), \text{ supp}(f) \text{ is compact} \}$$

denote the set of k–times continuously differentiable functions on $\mathbb{R}^n$ with compact support. For the case $k = 0$ we also use the notation $C_c(\mathbb{R}^n)$ to denote $C^0_c(\mathbb{R}^n)$.

Similarly, for any $1 \leq p < \infty$ one can define

$$L^p(\mathbb{R}^n) := \left\{ \varphi : \mathbb{R}^n \to \mathbb{R} : \int_{\mathbb{R}^n} |\varphi(\boldsymbol{x})|^p \, d\boldsymbol{x} < \infty \right\},$$

and

$$L^\infty(\mathbb{R}^n) := \left\{ \varphi : \mathbb{R}^n \to \mathbb{R} : \text{esssup}_{\boldsymbol{x} \in \mathbb{R}^n} \left| \varphi(\boldsymbol{x}) \right| < \infty \right\}$$

as well as *local versions of these spaces.* The notion of *essential supremum* (esssup) pertains for measure theory. When a given function is continuous, which is what we will mostly deal with, this is nothing but the normal supremum.

LEMMA 4.7 ($C_c(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$). *Let $f \in C_c(\mathbb{R}^n)$. Then $f \in L^\infty(\mathbb{R}^n)$.*

PROOF.

□

DEFINITION 4.8 (Convolution). Let $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in C_c(\mathbb{R}^n)$. The *convolution* of φ and f is the function $\varphi * f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(\varphi * f)(\boldsymbol{x}) = \int_{\mathbb{R}^n} \varphi(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}.$$

LEMMA 4.9 (Properties of the convolution). *Let $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in C_c(\mathbb{R}^n)$.*

(i) *The assumptions on φ and f ensure that the convolution is well-defined, i.e.,*

$$|(\varphi * f)(\boldsymbol{x})| < \infty \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$

(ii) *The convolution is commutative:*

$$\varphi * f = f * \varphi.$$

(iii) *If $\varphi \in L^1(\mathbb{R}^n)$, then $\varphi * f \in L^\infty(\mathbb{R}^n)$.*

(iv) *More generally, if $\varphi \in L^p(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$ with $p, q \in [1, \infty]$, then $\varphi * f \in L^r(\mathbb{R}^n)$ where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Also*

$$\|\varphi * f\|_{L^r(\mathbb{R}^n)} \leq \|\varphi\|_{L^p(\mathbb{R}^n)} \|f\|_{L^q(\mathbb{R}^n)}.$$

PROOF.

□

THEOREM 4.10 (Solution of Poisson's equation in $\mathbb{R}^n$). *Let $f \in C^2_c(\mathbb{R}^n)$ be twice continuously differentiable with compact support. Define*

$$u := \Phi * f.$$

Then $u \in C^2(\mathbb{R})$ *and* u *satisfies*

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

PROOF.

□

LEMMA 4.11 (Average of a function over the surface of a ball). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$. Then*

$$\fint_{\partial B_\varepsilon(\boldsymbol{x}_0)} g(\boldsymbol{z})\, dS(\boldsymbol{z}) \to g(\boldsymbol{x}_0) \quad \textit{as} \quad \varepsilon \to 0.$$

PROOF.

□

4.2.1. Fundamental Solution in $\mathbb{R}$.

THEOREM 4.12 (Solution of Poisson's equation in $\mathbb{R}$). *Let $f \in C_c^2(\mathbb{R})$ be twice continuously differentiable with compact support. Define*

$$\Phi(x) := \begin{cases} x & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases} \tag{4.2}$$

Define $u := \Phi * f$. Then $u \in C^2(\mathbb{R})$ and u satisfies

$$-u''(x) = f(x), \quad x \in \mathbb{R}. \tag{4.3}$$

We call Φ the fundamental solution of Poisson's equation in $\mathbb{R}$.

PROOF.

□

4.3. The Poincaré Inequality

THEOREM 4.13 (Poincaré inequality). *There exists a constant $C > 0$ such that*

$$\int_a^b |f(x) - \overline{f}|^2\,dx \le C^2 \int_a^b |f'(x)|^2\,dx$$

for all $f \in C^1([a,b])$, where $\overline{f}$ denotes the average of f over $[a,b]$:

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can write the Poincaré inequality in terms of L^2–norms as

$$\|f - \overline{f}\|_{L^2([a,b])} \le C\|f'\|_{L^2([a,b])}. \tag{4.4}$$

PROOF OF POINCARE'S INEQUALITY ON $[a,b]$.

□

THEOREM 4.14 (Poincaré inequality for functions that vanish on the boundary). *There exists a constant* $C > 0$ *such that*

$$\int_a^b |f(x)|^2\,dx \le C^2 \int_a^b |f'(x)|^2\,dx$$

for all $f \in C^1([a,b])$ *satisfying* $f(a) = f(b) = 0$. *We can write this in terms of* L^2*–norms as*

$$\|f\|_{L^2([a,b])} \le C\,\|f'\|_{L^2([a,b])}.$$

PROOF.

□

THEOREM 4.15 (The Poincaré inequality in higher dimensions). *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded. There exists a constant* $C > 0$ *such that*

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}$$

for all $f \in C^1(\overline{\Omega})$ *satisfying* $f = 0$ *on* $\partial\Omega$.

PROOF.

□

4.4. Poisson's Equation in $\Omega \subset \mathbb{R}^n$

4.4.1. Existence.

THEOREM 4.16 (Existence for Poisson's equation in general domains). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and connected with smooth boundary. Let $f \in C^1(\Omega)$ be bounded and $g \in C(\partial\Omega)$. Then there exists at least one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

PROOF.

□

4.4.2. Energy Method: Uniqueness and Continuous Dependence.

THEOREM 4.17 (Uniqueness for Poisson's equation). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with smooth boundary. There exists at most one solution $u \in C^2(\overline{\Omega})$ of the Dirichlet problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$.

PROOF.

□

DEFINITION 4.18 (H_0^1 and H^1 norms). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f \in C^1(\overline{\Omega})$.

(i) The H_0^1–norm of f is defined by

$$\|f\|_{H_0^1(\Omega)} := \|\nabla f\|_{L^2(\Omega)} = \left(\int_\Omega |\nabla f(\boldsymbol{x})|^2 \, d\boldsymbol{x}\right)^{1/2}.$$

It can be shown that $\|\cdot\|_{H_0^1(\Omega)}$ is a norm on $\{f \in C^1(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\}$.

(ii) The H^1–norm of f is defined by

$$\begin{aligned}\|f\|_{H^1(\Omega)} &:= \left(\|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)}\right)^{1/2} \\ &= \left(\int_\Omega |f(\boldsymbol{x})|^2 \, d\boldsymbol{x} + \int_\Omega |\nabla f(\boldsymbol{x})|^2 \, d\boldsymbol{x}\right)^{1/2}.\end{aligned}$$

It can be shown that $\|\cdot\|_{H^1(\Omega)}$ is a norm on $C^1(\overline{\Omega})$.

THEOREM 4.19 (Continuous dependence on data for Poisson's equation). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $f_1, f_2 \in C(\overline{\Omega})$. Let $u_1 \in C^2(\overline{\Omega})$ satisfy*

$$\begin{aligned} -\Delta u_1 &= f_1 \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and $u_2 \in C^2(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u_2 &= f_2 \quad \text{in } \Omega, \\ u_2 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then there exists a constant $C > 0$ such that

$$\|u_1 - u_2\|_{H_0^1(\Omega)} \leq C \|f_1 - f_2\|_{L^2(\Omega)}.$$

In simpler terms, this theorem says that if f_1 is close to f_2 (in the L^2–norm), then u_1 is close to u_2 (in the H_0^1–norm).

PROOF.

$\square$

4.5. Variational Formulation of Poisson's Equation

DEFINITION 4.20 (Weak solutions of Poisson's equation). We say that $u \in V$ is a *weak solution* of Poisson's equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

with $f \in C(\overline{\Omega})$ if

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\varphi(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{for all } \varphi \in V. \tag{4.6}$$

The functions φ are called *test functions.*

THEOREM 4.21 (Relation between the weak and classical formulations).

(i) *If* $u \in C^2(\overline{\Omega})$ *is a classical solution of Poisson's equation* (4.5)*, then it is a weak solution.*

(ii) *If* u *is a weak solution of Poisson's equation* (4.5) *and if in addition* $u \in C^2(\overline{\Omega})$*, then it is a classical solution.*

$$\boxed{\begin{array}{rcl} \text{classical formulation} & \Longrightarrow & \text{weak formulation} \\ \text{weak formulation} + \text{regularity} & \Longrightarrow & \text{classical formulation} \end{array}}$$

PROOF OF THE RELATIONSHIP BETWEEN THE WEAK AND CLASSICAL FORMULATIONS.

$\square$

DEFINITION 4.22 (Dirichlet energy). The *Dirichlet energy* is the function $E : V \to \mathbb{R}$ defined by

$$E[v] = \frac{1}{2}\int_\Omega |\nabla v(\boldsymbol{x})|^2 \, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x}.$$

E is a function of a function and we also refer to it as a *functional* or an *energy functional* or simply an *energy*.

THEOREM 4.23 (Dirichlet's principle: Minimising E is equivalent to solving Poisson's equation). *Let $u \in V$. The following are equivalent:*

(i) *u is a minimiser of E, i.e.,*

$$E[u] = \min_{v \in V} E[v].$$

(ii) *u is a weak solution of Poisson's equation* (4.5)*, i.e.,*

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\varphi(\boldsymbol{x})\, d\boldsymbol{x} \quad \textit{for all } \varphi \in V. \tag{4.7}$$

PROOF.

$\square$

COROLLARY 4.24 (C^2 minimisers of E satisfy Poisson's equation). *If $u \in C^2(\overline{\Omega}) \cap V$ is a minimiser of E, then u is a classical solution of Poisson's equation* (4.5).

CHAPTER 5

Laplace's Equation

5.1. Mean-Value Formulas

THEOREM 5.1 (Mean-Value Formulas). *Let $\Omega \subset \mathbb{R}^n$ be open. If $u \in C^2(\Omega)$ is harmonic in Ω, then*

$$u(\boldsymbol{x}) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dS(\boldsymbol{y}) = \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y}$$

for each ball $B_r(\boldsymbol{x}) \subset \Omega$. Therefore $u(\boldsymbol{x})$ equals the average of u over any sphere and over any ball in Ω centred at $\boldsymbol{x}$.

PROOF.

□

THEOREM 5.2 (Mean-Value Formula $\implies$ Harmonic). *Let $\Omega \subset \mathbb{R}^n$ be open. If $u \in C^2(\Omega)$ satisfies*

$$u(\boldsymbol{x}) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dS(\boldsymbol{y}) = \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y}$$

for each ball $B_r(\boldsymbol{x}) \subset \Omega$, then u is harmonic in Ω.

PROOF.

□

5.2. Maximum Principles

DEFINITION 5.3 (Open and closed subsets). Let $\Omega \subseteq \mathbb{R}^n$. We say that $U \subseteq \Omega$ is an *open subset of* Ω (or is *relatively open in* Ω) if $U = \Omega \cap \mathcal{O}$ for some open set $\mathcal{O} \subseteq \mathbb{R}^n$. A set $V \subseteq \Omega$ is a *closed subset of* Ω (or is *relatively closed in* Ω) if $V = \Omega \cap \mathcal{C}$ for some closed set $\mathcal{C} \subseteq \mathbb{R}^n$.

DEFINITION 5.4 (Connected sets). A set $\Omega \subseteq \mathbb{R}^n$ is *disconnected* if it can be written as the union of two disjoint nonempty open subsets of Ω. Otherwise it is *connected*.

LEMMA 5.5 (Subsets of connected sets). *Let $\Omega \subseteq \mathbb{R}^n$. The following are equivalent:*

(i) Ω *is connected.*

(ii) *The only subsets of* Ω *that are both open and closed subsets are* Ω *and the empty set.*

PROOF.

□

THEOREM 5.6 (Maximum Principles). *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded and connected. Let* $u : \overline{\Omega} \to \mathbb{R}$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *be harmonic in* Ω.

(i) *Weak maximum principle: u attains its maximum on the boundary of* Ω*, i.e.,*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) *Strong maximum principle: If u attains its maximum in* Ω*, then u is constant, i.e., if there exists* $\boldsymbol{x}_0 \in \Omega$ *such that*

$$u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u$$

then u is constant.

PROOF.

□

REMARK 5.7 (Minimum Principles). Harmonic functions also satisfy *minimum principles* in open sets, i.e., if u is harmonic then it attains its minimum on the boundary of Ω,

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

and if it also attains its minimum in Ω, then u is constant.

PROOF.

□

THEOREM 5.8 (Uniqueness for Poisson's equation). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. There exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$.

PROOF.

□

5.3. Maximum Principles for General Elliptic PDEs

DEFINITION 5.9 (Subharmonic and superharmonic functions). Let $\Omega \subseteq \mathbb{R}^n$ be open. We say that $u \in C^2(\Omega)$ is *subharmonic in* Ω if $-\Delta u(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x} \in \Omega$. We say that $u \in C^2(\Omega)$ is *superharmonic in* Ω if $-\Delta u(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \Omega$.

THEOREM 5.10 (Weak maximum principle for subharmonic functions). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

(i) *If u is subharmonic, then it satisfies the weak maximum principle*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) *If u is superharmonic, then it satisfies the weak minimum principle*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Moreover, subharmonic functions satisfy the strong maximum principle and superharmonic functions satisfy the strong minimum principle.

PROOF.

□

THEOREM 5.11 (Maximum principles for general elliptic PDEs with $c = 0$). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected and let $f \in C(\Omega)$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $Lu = f$, where L is a linear second-order elliptic operator of the form*

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} b_j u_{x_j} = -A : D^2 u + \boldsymbol{b} \cdot \nabla u$$

where a_{ij} and b_j are continuous functions on Ω, and A is symmetric and uniformly positive definite, which means that $a_{ij}(\boldsymbol{x}) = a_{ji}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ and that there exists a constant $\alpha > 0$ such that $\boldsymbol{y}^T A(\boldsymbol{x}) \boldsymbol{y} \geq \alpha |\boldsymbol{y}|^2$ for all $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$. Assume that $f \leq 0$.

(i) *Weak maximum principle: u attains its maximum on the boundary of Ω, i.e.,*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) *Strong maximum principle: If u attains its maximum in Ω, then u is constant, i.e., if there exists $\boldsymbol{x}_0 \in \Omega$ such that*

$$u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u$$

then u is constant in Ω.

Similarly, if $f \geq 0$, then u satisfies weak and strong minimum principles. In particular, if $f = 0$, then u satisfies weak and strong maximum and minimum principles.

5.4. Regularity of Harmonic Functions

THEOREM 5.12 (Regularity of Harmonic Functions). *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^2(\Omega)$ be harmonic. Then*

(i) $u \in C^\infty(\Omega)$,
(ii) *u is analytic in Ω, which means that u is infinitely differentiable and, for all $\boldsymbol{x}_0 \in \Omega$, the Taylor series of u about $\boldsymbol{x}_0$ converges to u in some neighbourhood of $\boldsymbol{x}_0$.*

PROOF OF THE REGULARITY OF HARMONIC FUNCTIONS.

□

THEOREM 5.13 (Liouville's Theorem). *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded harmonic function. Then u is constant.*

PROOF.

□

CHAPTER 6

The Heat Equation

DEFINITION 6.1 (Linear, second-order, parabolic PDEs). Let $\Omega \subseteq \mathbb{R}^n$ be open, $T > 0$, and $a_{ij}, b_j, c : \Omega \times (0,T) \to \mathbb{R}$ for $i, j \in \{1, \dots, n\}$. Let A be the matrix–valued function defined by $[A(\boldsymbol{x},t)]_{ij} = a_{ij}(\boldsymbol{x},t)$ and $\boldsymbol{b}$ be the vector–valued function defined by $[\boldsymbol{b}(\boldsymbol{x},t)]_j = b_j(\boldsymbol{x},t)$. Define the linear, second-order differential operator L by

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} b_j u_{x_j} + cu = -A : D^2 u + \boldsymbol{b} \cdot \nabla u + cu,$$

for $u : \Omega \times (0,T) \to \mathbb{R}$. PDEs of the form $u_t(\boldsymbol{x},t) + Lu(\boldsymbol{x},t) = f(\boldsymbol{x},t)$ are called *parabolic* if $A(\boldsymbol{x},t)$ is symmetric and uniformly positive definite, which means that $a_{ij}(\boldsymbol{x},t) = a_{ji}(\boldsymbol{x},t)$ for all $\boldsymbol{x} \in \Omega$, $t \in (0,T)$ and that there exists a constant $\alpha > 0$ such that $\boldsymbol{y}^T A(\boldsymbol{x},t)\boldsymbol{y} \geq \alpha |\boldsymbol{y}|^2$ for all $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$, $t \in (0,T)$. In particular, for fixed t, L is an elliptic operator.

6.1. Fourier Series and the Heat Equation in $\mathbb{R}/2\pi\mathbb{Z}$

THEOREM 6.2 (Fourier series). *Let $L > 0$ and $v \in L^2([0,L])$. Define $v_N \in L^2([0,L])$ by*

$$v_N(x) = \sum_{n=-N}^{N} \hat{v}_n e^{i\frac{2\pi n}{L}x}$$

where the Fourier coefficients $\hat{v}_n$ are defined by

$$\hat{v}_n = \frac{1}{L}\int_0^L v(x) e^{-i\frac{2\pi n}{L}x}\,dx. \tag{6.1}$$

Then v_N converges to v as $N \to \infty$ in the L^2–norm, i.e.,

$$\lim_{N\to\infty} \|v - v_N\|_{L^2([0,L])} = 0.$$

We write

$$v(x) = \sum_{n=-\infty}^{\infty} \hat{v}_n e^{i\frac{2\pi n}{L}x} \tag{6.2}$$

and call the right-hand side the Fourier series of v.

6.2. Fourier Transform and the Heat Equation in $\mathbb{R}^n$

6.2.1. The Fourier Transform.

DEFINITION 6.3 (The Fourier Transform). Let $v \in L^1(\mathbb{R}^n)$. We define its *Fourier transform* $\hat{v} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{v}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} v(\boldsymbol{x}) e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}} \, d\boldsymbol{x} \tag{6.3}$$

and its *inverse Fourier transform* $\check{v}:\mathbb{R}^n\to\mathbb{C}$ by

$$\check{v}(\boldsymbol{x})=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}v(\boldsymbol{\xi})e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}\,d\boldsymbol{\xi}. \tag{6.4}$$

THEOREM 6.4 (Properties of the Fourier transform (extended)).

(i) $\hat{v}\in L^\infty(\mathbb{R}^n)$. *Moreover,*

$$\|\hat{v}\|_{L^\infty(\mathbb{R}^n)}\le\|v\|_{L^1(\mathbb{R}^n)}.$$

(ii) $\hat{v}(\boldsymbol{\xi})$ *is uniformly continuous on* $\mathbb{R}^n$.

(iii) $\mathcal{F}:L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$ *is linear, i.e. for any* $u,v\in L^1(\mathbb{R}^n)$ *and any* $\alpha,\beta\in\mathbb{C}$ *we have that*

$$\left(\widehat{\alpha u+\beta v}\right)(\boldsymbol{\xi})=\mathcal{F}\left(\alpha u+\beta v\right)(\boldsymbol{\xi})=\alpha\mathcal{F}(u)(\boldsymbol{\xi})+\beta\mathcal{F}(v)(\boldsymbol{\xi})=\alpha\hat{u}(\boldsymbol{\xi})+\beta\hat{v}(\boldsymbol{\xi}).$$

(iv) *For a fixed* $\boldsymbol{a}\in\mathbb{R}^n$ *and* $u\in L^1(\mathbb{R}^n)$ *we have that* $u_{\boldsymbol{a}}(\boldsymbol{x})=u(\boldsymbol{x}-\boldsymbol{a})$ *is a function in* $L^1(\mathbb{R}^n)$ *and*

$$\hat{u_{\boldsymbol{a}}}(\boldsymbol{\xi})=\hat{u}(\boldsymbol{\xi})\,e^{-i\boldsymbol{\xi}\cdot\boldsymbol{a}}.$$

(v) *For a fixed* $\lambda>0$ *and* $u\in L^1(\mathbb{R}^n)$ *we have that* $u_\lambda(\boldsymbol{x})=\lambda^n u(\lambda\boldsymbol{x})$ *is a function in* $L^1(\mathbb{R}^n)$ *such that* $\|u\|_{L^1(\mathbb{R}^n)}=\|u_\lambda\|_{L^1(\mathbb{R}^n)}$ *and*

$$\hat{u_\lambda}(\boldsymbol{\xi})=\hat{u}\left(\frac{\boldsymbol{\xi}}{\lambda}\right).$$

(vi) *For a given multi-index*

$$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$$

we denote by $|\boldsymbol{\alpha}|=\sum_{j=1}^n\alpha_j$. *If* u *and* $\frac{\partial^{|\boldsymbol{\beta}|}u}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}\in L^1(\mathbb{R}^n)$ *for any multi-index* $\boldsymbol{\beta}$ *with* $\left|\boldsymbol{\beta}\right|\le\boldsymbol{\alpha}$ *then*

$$\mathcal{F}\left(\frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}\right)(\boldsymbol{\xi})=i^{|\boldsymbol{\alpha}|}\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}\hat{u}(\boldsymbol{\xi}).$$

(vii) *For a given multi-index*

$$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$$

we have that if u *and* $x_1^{\beta_1}\dots x_n^{\beta_n}u(\boldsymbol{x})\in L^1(\mathbb{R}^n)$ *for any multi-index* $\boldsymbol{\beta}$ *with* $\left|\boldsymbol{\beta}\right|\le\boldsymbol{\alpha}$ *then*

$$\mathcal{F}\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}u(\boldsymbol{x})\right)(\boldsymbol{\xi})=\frac{\partial^{|\boldsymbol{\alpha}|}\hat{u}(\boldsymbol{\xi})}{\partial\xi_1^{\alpha_1}\dots\partial\xi_n^{\alpha_n}}.$$

(viii) *The notion of convolution: For any* $u,v\in L^1(\mathbb{R}^n)$ *we can define*

$$u*v(\boldsymbol{x})=\int_{\mathbb{R}^n}u(\boldsymbol{x}-\boldsymbol{y})v(\boldsymbol{y})d\boldsymbol{y}=\int_{\mathbb{R}^n}u(\boldsymbol{y})v(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}.$$

(ix) *The inversion formula: If u and $\hat{u}$ belong to $L^1(\mathbb{R}^n)$ then*

$$u(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{u}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} d\boldsymbol{\xi}.$$

Denoting by

$$\check{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(\boldsymbol{x})\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} d\boldsymbol{x}$$

the inversion formula can be written as

$$u = \check{\hat{u}}.$$

(x) *The Fourier transform can be extended to a linear operator*

$$\mathcal{F} : L^p\left(\mathbb{R}^n\right) \to L^q\left(\mathbb{R}^n\right)$$

where $p \in [1,2]$ and q is its Hölder conjugate, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1$$

(in particular $q \in [2,\infty]$). When $p = q = 2$ we find that

$$\mathcal{F} : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right).$$

(xi) *Plancherel's identity: For any $u \in L^2(\mathbb{R}^n)$ we have that $\hat{u} \in L^2(\mathbb{R}^n)$ and*

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Moreover, if $u, v \in L^2(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} u(\boldsymbol{x})\, \overline{v}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \hat{u}(\boldsymbol{\xi})\, \overline{\hat{v}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

(xii) *The Fourier transform is unique: If $u, v \in L^p(\mathbb{R}^n)$ with $p \in [1,2]$ are such that $\hat{u}(\boldsymbol{\xi}) = \hat{v}(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$ then $u \equiv v$.*

6.2.2. The Fundamental Solution of the Heat Equation.

DEFINITION 6.5 (Fundamental solution of the heat equation). The *Fundamental solution of the heat equation in* $\mathbb{R}^n$ is the function $\Phi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ defined by

$$\Phi(\boldsymbol{x}, t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}}.$$

THEOREM 6.6 (Solution of the heat equation in $\mathbb{R}^n$). *Let $k > 0$ and $g \in C(\mathbb{R}^n)$ be bounded. Define $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by*

$$u(\boldsymbol{x}, t) := \frac{1}{(4\pi k t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4kt}} g(\boldsymbol{y})\, d\boldsymbol{y} = \Phi * g \tag{6.5}$$

where Φ is the fundamental solution of the heat equation in $\mathbb{R}^n$. Then

(i) *u is infinitely differentiable: $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$;*
(ii) *u satisfies the heat equation: $u_t = k\Delta u$ in $\mathbb{R}^n \times (0,\infty)$;*
(iii) *u has initial value g: For each point $\boldsymbol{x}_0 \in \mathbb{R}^n$*

$$\lim_{\substack{(\boldsymbol{x},t)\to(\boldsymbol{x}_0,0)\\ \boldsymbol{x}\in\mathbb{R}^n,\, t>0}} u(\boldsymbol{x}, t) = g(\boldsymbol{x}_0).$$

PROOF.

$\square$

REMARK 6.7 (Solution formula for the heat equation with source term). Consider the heat equation on $\mathbb{R}^n$ with a source term:

$$\begin{aligned} u_t - k\Delta u &= f \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u &= g \quad \text{for } t = 0. \end{aligned}$$

This is satisfied by

$$u(\boldsymbol{x}, t) = \int_{\mathbb{R}^n} \Phi(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y} + \int_0^t \int_{\mathbb{R}^n} \Phi(\boldsymbol{x} - \boldsymbol{y}, t - s) f(\boldsymbol{y}, s)\, d\boldsymbol{y}\, ds.$$

6.3. The Energy Method

THEOREM 6.8 (Uniqueness for the heat equation). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with smooth boundary. Let $k > 0$, $T > 0$. There exists at most one smooth solution $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ of the heat equation*

$$\begin{aligned} u_t - k\Delta u &= f \quad \text{in } \Omega \times (0,T], \\ u &= g \quad \text{on } \partial\Omega \times [0,T], \\ u &= u_0 \quad \text{on } \Omega \times \{0\}. \end{aligned}$$

PROOF.

□

LEMMA 6.9 (The Grönwall inequality). *Let* $E : [0,\infty) \to \mathbb{R}$ *be a continuously differentiable function satisfying* $E' \le -\lambda E$ *for some constant* $\lambda \in \mathbb{R}$. *Then* $E(t) \le e^{-\lambda t}E(0)$ *for all* $t \ge 0$.

PROOF.

□

THEOREM 6.10 (Sobolev Embedding Theorem). *Let* $f \in C^1([a,b])$.

(i) *For all* $x, y \in [a,b]$,

$$|f(y) - f(x)| \le \|f'\|_{L^2([a,b])} |y - x|^{\frac{1}{2}}.$$

In other words, f *is Hölder continuous with exponent* $1/2$.

(ii) *Sobolev inequality: There exists a constant $C > 0$ such that*

$$\|f\|_{L^\infty([a,b])} \le C\|f\|_{H^1([a,b])}$$

where $\|f\|_{H^1([a,b])} = \left(\|f\|^2_{L^2([a,b])} + \|f'\|^2_{L^2([a,b])}\right)^{\frac{1}{2}}$.

THEOREM 6.11 (Asymptotic behaviour of the heat equation with periodic BCs). *Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be smooth and 2π–periodic in x, i.e., $u(x+2\pi, t) = u(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$. Let u satisfy*

$$\begin{aligned} u_t - ku_{xx} = 0 &\quad \text{for } (x,t) \in (0,2\pi) \times (0,\infty), \\ u(x,0) = u_0(x) &\quad \text{for } x \in (0,2\pi), \end{aligned}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a smooth 2π–periodic function. Let $\overline{u}_0 = \frac{1}{2\pi}\int_0^{2\pi} u_0(x)\,dx$ denote the average value of u_0. Then $u \to \overline{u}_0$ in $L^\infty([0,2\pi])$ as $t \to \infty$:

$$\lim_{t\to\infty} \|u(\cdot,t) - \overline{u}_0\|_{L^\infty([0,2\pi])} = 0.$$

In other words, the temperature converges uniformly to the average initial temperature as $t \to \infty$.

PROOF.

□

THEOREM 6.12 (Asymptotic behaviour of the heat equation with time independent data). *Let* $\Omega \subset \mathbb{R}^n$ *be open, bounded and connected with smooth boundary. Let* $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ *be a smooth function satisfying*

$$\begin{aligned} u_t(\boldsymbol{x},t) - k\Delta u(\boldsymbol{x},t) &= f(\boldsymbol{x}) \quad \textit{for } (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ u(\boldsymbol{x},t) &= g(\boldsymbol{x}) \quad \textit{for } (\boldsymbol{x},t) \in \partial\Omega \times [0,\infty), \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \quad \textit{for } \boldsymbol{x} \in \Omega, \end{aligned}$$

where f, g, u_0 *are given smooth functions. Let* $v : \overline{\Omega} \to \mathbb{R}$ *be a smooth, time independent solution of the same equation:*

$$\begin{aligned} -k\Delta v(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \textit{for } \boldsymbol{x} \in \Omega, \\ v(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \textit{for } \boldsymbol{x} \in \partial\Omega. \end{aligned}$$

Then

$$\lim_{t\to\infty} \|u - v\|_{L^2(\Omega)} = 0.$$

In other words, if the source term f *and boundary data* g *are independent of time, then the solution of the heat equation converges in the* L^2*–norm to the solution of Poisson's equation as* $t \to \infty$*.*

6.4. Maximum Principles

DEFINITION 6.13 (Parabolic domain and parabolic boundary). We define

$$\Omega_T = \Omega \times (0, T]$$

and we refer to Ω_T as a *cylinder*. Note that Ω_T includes the top of the cylinder $\Omega \times \{t = T\}$ but not the bottom $\Omega \times \{t = 0\}$. The *parabolic boundary* of Ω_T is defined by

$$\Gamma_T = \overline{\Omega}_T \setminus \Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T]),$$

which is the bottom and sides of the cylinder Ω_T but not the interior or the top.

DEFINITION 6.14 (The function space C_1^2). Define $C_1^2(\Omega_T)$ to be the space of functions on Ω_T that are once continuously-differentiable in time and twice continuously-differentiable in space:

$$C_1^2(\Omega_T) = \left\{u : \Omega_T \to \mathbb{R} : u, u_t, u_{x_i}, u_{x_i x_j} \in C(\Omega_T) \ \forall\, i, j \in \{1, \dots, n\}\right\}.$$

Recall the following facts:

- A matrix $A \in \mathbb{R}^{m\times m}$ is *negative semi-definite* if
$$\boldsymbol{y} \cdot A\boldsymbol{y} \le 0 \quad \forall\ \boldsymbol{y} \in \mathbb{R}^m.$$
- Let $U \subseteq \mathbb{R}^m$ be open and let $g \in C^2(U)$. Suppose that $\boldsymbol{y}_0 \in U$ is a local maximum point of g. Then
$$\nabla g(\boldsymbol{y}_0) = \boldsymbol{0}, \qquad D^2 g(\boldsymbol{y}_0) \text{ is negative semi-definite}$$
where $D^2 g$ is the matrix of second partial derivatives of g, which has components $[D^2 g]_{ij} = g_{y_i y_j}$, $i, j \in \{1, \dots, m\}$.

THEOREM 6.15 (Weak maximum principle for the heat equation). *Let* $k>0$ *and let* $\Omega\subset\mathbb{R}^n$ *be open and bounded. Let* $u:\overline{\Omega_T}\to\mathbb{R}$, $u\in C^2_1(\Omega_T)\cap C(\overline{\Omega_T})$.

(i) *If*

$$u_t-k\Delta u\le 0 \text{ in } \Omega_T,$$

then

$$\max_{\overline{\Omega_T}} u=\max_{\Gamma_T} u.$$

In other words, if u *is a solution of the heat equation* $u_t-k\Delta u=f$ *in* Ω_T *with* $f\le 0$, *then* u *attains its maximum on the parabolic boundary* Γ_T.

(ii) *If*

$$u_t-k\Delta u\ge 0 \text{ in } \Omega_T,$$

then

$$\min_{\overline{\Omega_T}} u=\min_{\Gamma_T} u.$$

In other words, if u *is a solution of the heat equation* $u_t-k\Delta u=f$ *in* Ω_T *with* $f\ge 0$, *then* u *attains its minimum on the parabolic boundary* Γ_T.

(iii) *If*

$$u_t-k\Delta u=0 \text{ in } \Omega_T,$$

then

$$\max_{\overline{\Omega_T}} u=\max_{\Gamma_T} u,\qquad \min_{\overline{\Omega_T}} u=\min_{\Gamma_T} u.$$

PROOF.

□

THEOREM 6.16 (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $u : \overline{\Omega_T} \to \mathbb{R}$, $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$. Assume additionally that Ω is connected.*

(i) *If $u_t - k\Delta u \leq 0$ in Ω_T, and if u attains its maximum over $\overline{\Omega_T}$ at a point $(\boldsymbol{x}_0, t_0) \in \Omega_T$, then u is constant in $\Omega_{t_0} = \Omega \times (0, t_0]$.*

(ii) *If $u_t - k\Delta u \geq 0$ in Ω_T, and if u attains its minimum over $\overline{\Omega_T}$ at a point $(\boldsymbol{x}_0, t_0) \in \Omega_T$, then u is constant in $\Omega_{t_0} = \Omega \times (0, t_0]$.*

CHAPTER 7

The Wave Equation

DEFINITION 7.1 (Linear, second-order, hyperbolic PDEs). Let $\Omega \subseteq \mathbb{R}^n$ be open, $T > 0$, and $a_{ij}, b_j, d : \Omega \times (0, T) \to \mathbb{R}$ for $i, j \in \{1, \ldots, n\}$. Let A be the matrix–valued function defined by $[A(\boldsymbol{x}, t)]_{ij} = a_{ij}(\boldsymbol{x}, t)$ and $\boldsymbol{b}$ be the vector–valued function defined by $[\boldsymbol{b}(\boldsymbol{x}, t)]_j = b_j(\boldsymbol{x}, t)$. Define the linear, second-order differential operator L by

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{j=1}^{n} b_j u_{x_j} + du = -A : D^2 u + \boldsymbol{b} \cdot \nabla u + du,$$

for $u : \Omega \times (0, T) \to \mathbb{R}$. PDEs of the form $u_{tt}(\boldsymbol{x}, t) + Lu(\boldsymbol{x}, t) = f(\boldsymbol{x}, t)$ are called *hyperbolic* if $A(\boldsymbol{x}, t)$ is symmetric and uniformly positive definite, which means that $a_{ij}(\boldsymbol{x}, t) = a_{ji}(\boldsymbol{x}, t)$ for all $\boldsymbol{x} \in \Omega$, $t \in (0, T)$ and that there exists a constant $\alpha > 0$ such that $\boldsymbol{y}^T A(\boldsymbol{x}, t)\boldsymbol{y} \geq \alpha |\boldsymbol{y}|^2$ for all $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$, $t \in (0, T)$. In particular, for fixed t, L is an elliptic operator.

7.1. The Wave Equation in ℝ

7.1.1. D'Alembert's Solution.

REMARK 7.2 (The non-homogeneous wave equation). Duhamel's principle can be used to show that the non-homogeneous wave equation

$$\begin{aligned} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= f(x,t) \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= h(x) \quad \text{for } x \in \mathbb{R}, \end{aligned}$$

is satisfied by

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

7.2. The Energy Method

THEOREM 7.3 (Conservation of energy). *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with smooth boundary. Let* $T > 0$. *Suppose that* $u \in C^2(\overline{\Omega} \times [0, T])$ *satisfies the following wave equation:*

$$\begin{aligned} u_{tt} - c^2 \Delta u = 0 &\quad \text{in } \Omega \times (0, T], \\ u = 0 &\quad \text{on } \partial\Omega \times [0, T], \\ u = g &\quad \text{on } \Omega \times \{0\}, \\ u_t = h &\quad \text{on } \Omega \times \{0\}, \end{aligned}$$

where $g, h : \Omega \to \mathbb{R}$. *Define the energy*

$$E(t) = \frac{1}{2} \int_\Omega u_t^2(\boldsymbol{x}, t)\, d\boldsymbol{x} + \frac{1}{2} c^2 \int_\Omega |\nabla u(\boldsymbol{x}, t)|^2\, d\boldsymbol{x}.$$

Then energy is conserved:

$$\frac{dE}{dt} = 0.$$

In other words, $E(t) = E(0)$ *for all* $t \geq 0$. *If we regard* Ω *as an elastic body, then* E *can be interpreted as the sum of its kinetic energy and its elastic potential energy.*

PROOF.

□

COROLLARY 7.4 (Uniqueness of $C^2\left(\overline{\Omega}\times[0,T]\right)$ solutions). *Let $\Omega\subset\mathbb{R}^n$ be open and bounded with smooth boundary. Let $T>0$. Consider the following wave equation:*

$$\begin{aligned} u_{tt}-c^2\Delta u &= 0 \quad \text{in } \Omega\times(0,T],\\ u &= 0 \quad \text{on } \partial\Omega\times[0,T],\\ u &= g \quad \text{on } \Omega\times\{0\},\\ u_t &= h \quad \text{on } \Omega\times\{0\}, \end{aligned}$$

where $g,h:\Omega\to\mathbb{R}$. Then there exists at most one $C^2\left(\overline{\Omega}\times[0,T]\right)$ solution to the above equation.

PROOF.

□

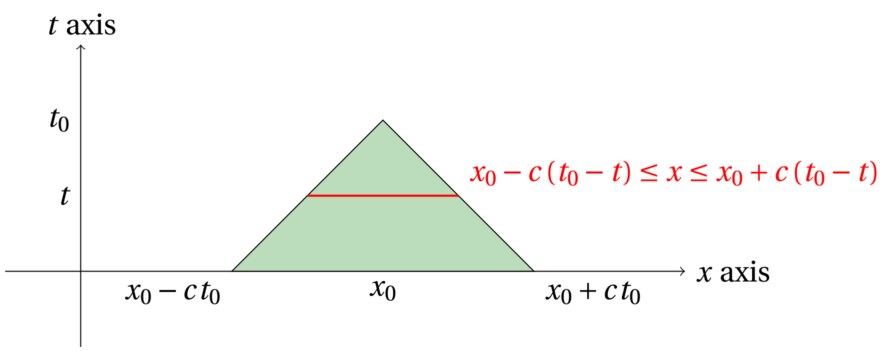
THEOREM 7.5 (Finite speed of propagation). *Let* $u \in C^2(\mathbb{R} \times [0,\infty))$ *satisfy*

$$u_{tt} = c^2 u_{xx} \quad \text{in } \mathbb{R} \times (0,\infty).$$

Fix $x_0 \in \mathbb{R}$, $t_0 > 0$. *Define*

$$\mathcal{T} = \{(x,t) \in \mathbb{R} \times [0,t_0] : -c(t_0 - t) \le x - x_0 \le c(t_0 - t)\}.$$

This is the triangle in the (x,t)*–plane with tip* (x_0, t_0) *and base* $[x_0 - ct_0, x_0 + ct_0] \times \{0\}$. *If* $u(x,0) = u_t(x,0) = 0$ *for* $x \in [x_0 - ct_0, x_0 + ct_0]$, *then* $u = 0$ *in* $\mathcal{T}$.



$\mathcal{T} = \{(x,t) \in \mathbb{R} \times [0,t_0] : -c(t_0 - t) \le x - x_0 \le c(t_0 - t)\}$