PARTIAL DIFFERENTIAL EQUATIONS III & V
REVISION CLASS SOLUTION

Question 1 *(Q2 – May 2024 exam).* We consider the following Cauchy problem for the scalar unknown function u that we aim to solve by the method of characteristics.

$$\begin{cases} 5 - x_1^2 \partial_{x_2} u(x_1,x_2) = 0, & (x_1,x_2) \in \mathbb{R}^2, \\ u(x_1,0) = 3, & x_1 \in \mathbb{R}. \end{cases} \tag{1}$$

a) Determine the leading vector field, the Cauchy datum and the Cauchy curve associated to this problem.
b) Find all the points on the Cauchy curve which are noncharacteristic.
c) Write down the ODE system for the characteristics and for the solution along the characteristics. Then solve this system.
d) Sketch a few characteristic curves.
e) Find the solution u to (1). Determine its maximal domain of definition.

Solution:

a) Rewriting the equation as

$$\begin{cases} 0\,\partial_{x_1} u(x_1,x_2) + x_1^2 \partial_{x_2} u(x_1,x_2) = 5, & (x_1,x_2) \in \mathbb{R}^2, \\ u(x_1,0) = 3, & x_1 \in \mathbb{R}. \end{cases}$$

we see that the leading vector field is $\vec{a}\,(x_1,x_2) = \left(0, x_1^2\right)$, the Cauchy curve is the x_1 axis given by

$$\{\gamma(s) := (s,0) : s \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and the Cauchy datum is $u\,(s,0) = 3$.

b) As $\gamma'(s) = (1,0)$ we see that the condition for being non-characteristic is

$$0 \neq \vec{a}\left(\gamma(s)\right) \cdot (0,-1) = -s^2,$$

which shows that all the points on the Cauchy curve, except $(0,0)$, are non-characteristic.

c) The ODE system is given by

$$\begin{cases} \partial_\tau x_1(\tau,s) = 0, \\ \partial_\tau x_2(\tau,s) = x_1^2(\tau,s), \\ \partial_\tau z(\tau,s) = 5 \end{cases}$$

with the initial datum

$$x_1(0,s) = s, \quad x_2(0,s) = 0, \quad z(0,s) = 3.$$

The solution is given by

$$x_1(\tau,s) = x_1(0,s) = s,$$

$$z(\tau,s) = 5\tau + z(0,s) = 3 + 5\tau,$$

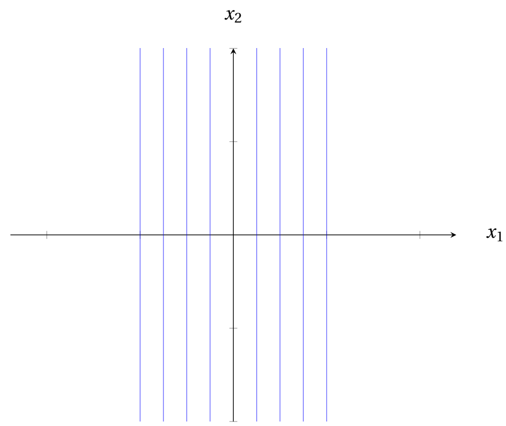
and since $\partial_\tau x_2(\tau,s) = x_1^2(\tau,s) = s^2$ we have that

$$x_2(\tau,s) = x_2(0,s) + s^2\tau = s^2\tau.$$

d) For a fixed $s \in \mathbb{R} \setminus \{0\}$ we find that the characteristic curves are given by

$$\tau \longmapsto (x_1(\tau, s), x_2(\tau, s)) = (s, s^2\tau).$$

These are lines parallel to the x_2-axis.



e) In order to find the solution we need to invert the map $(\tau, s) \to (x_1, x_2)$. We have that $x_1 = s$ and

$$\tau = \frac{x_2}{s^2} = \frac{x_2}{x_1^2}.$$

Note that $x_1 \neq 0$ since we do not consider the point $(0,0)$. We conclude that

$$u(x_1, x_2) = z(\tau(x_1, x_2), s(x_1, x_2)) = 3 + 5\frac{x_2}{x_1^2}.$$

The domain of this function is $\mathbb{R}^2 \setminus \{(0, x_2) : x_2 \in \mathbb{R}\}$.

Question 2 *(Q3 – May 2024 exam).* Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary, and suppose that $d \geq 2$.

a) Suppose that $u, v \in C^2(\overline{\Omega})$. Show that the following formula holds

$$\int_\Omega v\Delta u d\vec{x} + \int_\Omega \nabla v \cdot \nabla u d\vec{x} = \int_{\partial\Omega} v\partial_n u dS,$$

where ∇ stands for the gradient, Δ stands for the Laplace operator and we used the notation $\partial_n u = \nabla u \cdot \vec{n}$, with $\vec{n}$ being the outward pointing unit normal vector to $\partial\Omega$.

b) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that

$$\int_{\partial\Omega} \partial_n u dS = 0.$$

c) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that $\int_{\partial\Omega} u\partial_n u dS$ is nonnegative.

d) Suppose that $u, v : \Omega \to \mathbb{R}$ are both harmonic and $u, v \in C^2(\overline{\Omega})$. Show that

$$\int_{\partial\Omega} (u\partial_n v - v\partial_n u)\, dS = 0.$$

Solution:

a) We recall that for any two C^2 functions u, v

$$v\Delta u = v\mathrm{div}(\nabla u) = \mathrm{div}(v\nabla u) - \nabla v \cdot \nabla u.$$

Using the above with the divergence theorem we find that

$$\int_\Omega v\Delta u dx = \int_\Omega \operatorname{div}(v\nabla u)\, dx - \int_\Omega \nabla v\cdot\nabla u dx$$
$$= \int_{\partial\Omega} v\nabla u\cdot\vec{n} dS - \int_\Omega \nabla v\cdot\nabla u dx$$
$$= \int_{\partial\Omega} v\partial_n u dS - \int_\Omega \nabla v\cdot\nabla u dx,$$

which is the desired result.

b) Using the formula we found in a) with $v = 1$ and u we find that

$$0 + 0 = \int_{\partial\Omega} 1\, \partial_n u dS.$$

c) Using the formula we found in a) with $v = u$ we find that

$$0 + \int_\Omega |\nabla u|^2\, dx = \int_{\partial\Omega} u\partial_n u dS.$$

Consequently,

$$\int_{\partial\Omega} u\partial_n u dS = \int_\Omega |\nabla u|^2\, dx \geq 0.$$

d) Since u is harmonic the formula we found in a) states that

$$\int_\Omega \nabla u\cdot\nabla v dx = \int_{\partial\Omega} v\partial_n u dS.$$

Replacing u with v and using the fact that v is harmonic gives

$$\int_\Omega \nabla v\cdot\nabla u dx = \int_{\partial\Omega} u\partial_n v dS.$$

Subtracting these two equations gives the desired result.

Question 3 *(Q5 – May 2024 exam).* Let $\alpha \in \mathbb{R}$ and set $A^\alpha = (a^\alpha_{ij})^2_{i,j=1} \in \mathbb{R}^{2\times 2}$ to be the matrix $A^\alpha := \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$. For a given open set $\Omega \subseteq \mathbb{R}^2$ and $u \in C^2(\Omega)$, we define the differential operator

$$(\mathcal{L}^\alpha u)(\vec{x}) := -A^\alpha : D^2 u(\vec{x}) = -\sum_{i,j=1}^2 a^\alpha_{ij}\partial_{x_i}\partial_{x_j} u(\vec{x}),$$

where $D^2 u$ stands for the Hessian matrix of u.

a) Show that the matrix A^α is positive semi-definite if and only if $|\alpha| \leq 1$. Show that A^α is positive definite if and only if $|\alpha| < 1$.

b) Let Ω be open, bounded and connected with smooth boundary. Suppose that $|\alpha| < 1$ and $u : \Omega \to \mathbb{R}$ is a classical solution to

$$(\mathcal{L}^\alpha u)(\vec{x}) = 0,\ \ \vec{x} \in \Omega.$$

Explain why u attains both its minimum and maximum on $\partial\Omega$.

c) Now we set $\alpha = 1$. Find all those real numbers $c_1, c_2 \in \mathbb{R}$ for which the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$u(x_1, x_2) = c_1(x_1^2 + x_2^2) - c_2 x_1 x_2$$

is a solution to $\mathcal{L}^1 u = 0$.

d) Suppose that we are in the setting of the previous point (c). Show that u fails to satisfy either the strong minimum or the strong maximum principle (one of the two). [*Hint*: choose c_1, c_2 such that $u(x_1,x_2) \geq 0$ for all $(x_1,x_2) \in \mathbb{R}^2$. Find a particular bounded connected domain $\Omega \subset \mathbb{R}^2$, which is a sublevel set of u, i.e. $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 : \ u(x_1,x_2) < r\}$, for some $r > 0$. Deduce the failure of the strong minimum principle in this domain.
Remark: This is a corrected version of the problem which asked about the weak minimum/maximum principle.]

Solution:

a) A symmetric matrix is positive semi-definite if and only if all its eigenvalues are non-negative. It is positive definite if and only if all its eigenvalues are positive. For a 2×2 symmetric matrix A with eigenvalues λ_1 and λ_2 we know that

$$\det(A) = \lambda_1\lambda_2, \qquad \mathrm{tr}\,(A) = \lambda_1 + \lambda_2.$$

Consequently A is positive semi-definite if and only if $\det(A) \geq 0$ and $\mathrm{tr}(A) \geq 0$ and is positive definite if and only if $\det(A) > 0$ and $\mathrm{tr}(A) > 0$.

In our case $\det(A) = 1-\alpha^2$ and $\mathrm{tr}(A) = 2 > 0$. We conclude that if $|\alpha| < 1$ the matrix is positive definite and if $|\alpha| \leq 1$ the matrix is positive semi-definite.

b) Since $|\alpha| < 1$ our operator $\mathcal{L}^\alpha$ is an elliptic operator. Consequently, any classical solution u to

$$\mathcal{L}^\alpha u = 0$$

is both a super and sub-solution to our equation, which implies that it satisfies the weak minimum and maximum principles. As a result, since u is continuous on the closure of the domain and as such must attain a minimum and maximum on it, u attains both its minimum and maximum on $\partial\Omega$.

c) We have that

$$\mathcal{L}^1 u(x_1,x_2) = -\Delta u(x_1,x_2) - 2\partial_{x_1x_2} u(x_1,x_2)$$

for any $u \in C^2(\Omega)$. If $u(x_1,x_2) = c_1(x_1^2 + x_2^2) - c_2 x_1 x_2$ solves the equation $\mathcal{L}^1 u = 0$ we must have that

$$0 = -2c_1 - 2c_1 - 2(-c_2) = 2c_2 - 4c_1.$$

In other words

$$u(x_1,x_2) = c_1\left(x_1^2 + x_2^2 - 2x_1x_2\right) = c_1(x_1 - x_2)^2.$$

d) Following the hint, we see that if we choose $c_1 > 0$ we have that $u(x_1,x_2) \geq 0$ for all x_1, x_2. Choosing any open and bounded set that intersects the line $x_1 = x_2$ will give us an interior point $(\hat{x}_1, \hat{x}_1)$ for which we find that

$$u(\hat{x}_1, \hat{x}_1) = 0,$$

i.e. u attains a global minimum in an interior point but u is not constant.

Question 4 *(Q6 – May 2024 exam)*. Let $f : \mathbb{R} \to \mathbb{R}$ of class C^2 be given. Suppose that this is strongly convex, i.e. there exists $c_0 > 0$ such that $f''(x) \geq c_0$ for all $x \in \mathbb{R}$. Consider the following Cauchy problem for the unknown $u : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$

$$\begin{cases} \partial_t u(x,t) + \partial_x(f(u(x,t))) = 0, & (x,t) \in \mathbb{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

For $\varepsilon > 0$ we consider the following approximation of (2)

$$\begin{cases} \partial_t u^\varepsilon(x,t) + \partial_x(f(u^\varepsilon(x,t))) - \varepsilon\partial^2_{xx}u^\varepsilon(x,t) = 0, & (x,t) \in \mathbb{R}\times(0,+\infty), \\ u^\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

a) State Lax's entropy condition for weak solutions to the Cauchy problem (2).

b) We look for a solution to (3) in the form

$$u^\varepsilon(x,t) := v\left(\frac{x-\alpha t}{\varepsilon}\right), \tag{4}$$

for a given constant $\alpha \in \mathbb{R}$ and some given smooth enough function $v : \mathbb{R} \to \mathbb{R}$. Find the second order ODE that v needs to satisfy in order for the formula (4) to give a classical solution to (3).

c) Let $u_\ell, u_r \in \mathbb{R}$ be given, and we are looking for a solution to the ODE for v found in (b) with the additional assumptions

$$\lim_{s\to-\infty} v(s) = u_\ell; \quad \lim_{s\to+\infty} v(s) = u_r; \quad \lim_{s\to\pm\infty} v'(s) = 0.$$

Suppose that we find such a solution v. Compute the limit $\lim\limits_{\varepsilon\to 0} u^\varepsilon(x,t)$, in the case when $x \neq \alpha t$.

d) Suppose that we are in the setting of (c). Find an equation that α needs to satisfy, in terms of f and u_ℓ, u_r. [*Hint*: integrate the second oder ODE for v, then take limits $s \to \pm\infty$].

e) Suppose that $u_0(x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases}$ Suppose that $u_r < u_l$. Suppose that (3) has a classical solution in the form of (4), and v and α satisfy all the previously set and obtained properties. Conclude that $u^\varepsilon(x,t) \to u(x,t)$, as $\varepsilon \to 0$, almost everywhere, where u is the unique solution to (2) which satisfies Lax's entropy condition.

Solution:

a) Lax's entropy condition for weak solutions to the Cauchy problem is that every shock to the problem $(\sigma(t), t)$ satisfies

$$f'(u_r) < \dot{\sigma} < f'(u_l)$$

where u_ℓ and u_r are the left and right limits of the solution across the shock. Since f is strongly convex this condition is equivalent to $u_r < u_\ell$.

b) We have that

$$\partial_t u^\varepsilon(x,t) = -\frac{\alpha}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right),$$

$$\partial_x u^\varepsilon(x,t) = \frac{1}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right),$$

$$\partial_{xx} u^\varepsilon(x,t) = \frac{1}{\varepsilon^2} v''\left(\frac{x-\alpha t}{\varepsilon}\right).$$

Plugging this back in (3) gives us

$$-\frac{\alpha}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right) + \frac{1}{\varepsilon} f'\left(v\left(\frac{x-\alpha t}{\varepsilon}\right)\right) v'\left(\frac{x-\alpha t}{\varepsilon}\right) - \frac{1}{\varepsilon} v''\left(\frac{x-\alpha t}{\varepsilon}\right) = 0.$$

As $\frac{x-\alpha t}{\varepsilon}$ ranges over all points in $\mathbb{R}$ the ODE that v must satisfy is

$$v''(s) = -\alpha v'(s) + f'(v(s))v'(s), \qquad s \in \mathbb{R}.$$

c) We have that

$$\lim_{\varepsilon\to 0} u^{\varepsilon}(x,t) = \begin{cases} u_\ell, & x<\alpha t, \\ u_r, & x>\alpha t. \end{cases}$$

d) We notice that

$$-\alpha v'(s) + f'(v(s))v'(s) = \frac{d}{ds}\big(-\alpha v(s) + f(v(s))\big)$$

and consequently we can integrate the ODE of v to get that

$$v'(s) = -\alpha v(s) + f(v(s)) + C,$$

for any $s \in \mathbb{R}$. Taking s to ∞ in the above we find that

$$0 = -\alpha u_r + f(u_r) + C$$

Similarly, taking s to $-\infty$ yields

$$0 = -\alpha u_\ell + f(u_\ell) + C.$$

Subtracting the two equations we find that

$$\alpha(u_r - u_\ell) = f(u_r) - f(u_\ell).$$

e) We notice that for any $x \neq \alpha t$

$$u(x,t) = \lim_{\varepsilon\to 0} u^{\varepsilon}(x,t) = \begin{cases} u_\ell, & x<\alpha t, \\ u_r, & x>\alpha t. \end{cases}$$

solves (2) with the correct initial conditions. Moreover, on the shock curve $\sigma(t) = \alpha t$ we have that

$$\dot{\sigma} = \alpha = \frac{f(u_r) - f(u_\ell)}{u_r - u_\ell}.$$

The mean value theorem implies that we can find c between u_r and u_ℓ such that

$$\frac{f(u_r) - f(u_\ell)}{u_r - u_\ell} = f'(c)$$

Since f is strictly convex, f' is increasing and we conclude that

$$f'(u_r) < f'(c) < f'(u_\ell)$$

as we assumed that $u_r < u_\ell$. We conclude that the entropy condition is satisfied.

Question 5 *(Q7 – May 2024 exam)*. We consider the following Cauchy problem

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

We set

$$u_0(x) = \begin{cases} 0, & x<0, \\ 1, & 0<x<1, \\ 2, & 1<x<2, \\ x, & 2<x. \end{cases}$$

We aim to construct a unique entropy solution to this Cauchy problem.

a) Sketch the characteristic lines associated with the Cauchy problem and discuss about the need of shock curves and/or rarefaction waves.
b) Introduce the corresponding shocks and/or rarefaction waves.
c) Write down the candidate for the weak entropy solutions to (5).
d) Show that this solution is continuous everywhere if $t > 0$.

e) Show that the solution satisfies Lax's entropy condition.

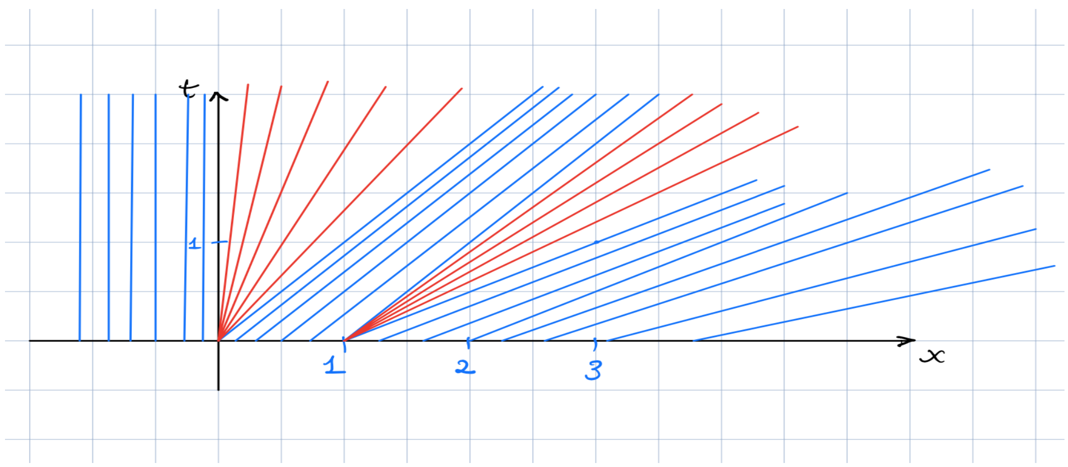
Solution:

a) For any interval I on which u_0 is continuous we get the characteristics curves are straight lines which start at $(s, 0)$ with $s \in I$ and have a slope $u_0(s)$. In particular we find that the characteristics are

$$x = s + u_0(s)t = \begin{cases} s, & s < 0, \\ s + t, & 0 < s < 1, \\ s + 2t, & 1 < s < 2, \\ s + st, & 2 < s, \end{cases}$$

As the slopes keep increasing when defined (which we could have said from the fact that u_0 is non-decreasing) we see that the characteristics will not cross, and there will be empty regions which they will not reach. In other words, there will be no shocks but we will need rarefaction waves to construct a unique entropy solution.

A sketch of the characteristics (in blue) and the rarefaction waves we'll need to add (in red) is given below.



b) We have two rarefaction regions: The wedge which is between the lines $x = 0$ and $x = t$ which meet at $(0, 0)$, and the wedge between the line $x = 1 + t$ and $x = 1 + 2t$ which meet at $(1, 0)$.

We will look for a solution of the form $g_1\left(\frac{x}{t}\right)$ with $g_1(0) = 0$ and $g_1(1) = 1$ for the first region and $g_2\left(\frac{x-1}{t}\right)$ with $g_2(1) = 1$ and $g_2(2) = 2$ for the second.

For $v(x, t) = h\left(\frac{x-x_0}{t}\right)$ to be a solution for our equation we must have that

$$-\frac{x - x_0}{t^2} h'\left(\frac{x - x_0}{t}\right) + \frac{h\left(\frac{x-x_0}{t}\right)}{t} h'\left(\frac{x - x_0}{t}\right) = 0,$$

which is equivalent to saying that

$$h'(z)\left(h(z) - z\right) = 0,$$

showing that $h(z) = z$ is a possible solution. Due to our initial conditions we see that it is indeed the correct solution, i.e. $g_1\left(\frac{x}{t}\right) = \frac{x}{t}$ and $g_2\left(\frac{x-1}{t}\right) = \frac{x-1}{t}$.

c) Following on our above discussion we identify the following regions:

<u>$x < 0$:</u> Here we can use characteristics to find that $u(x, t) = u_0(x) = 0$.

<u>$0 < x < t$:</u> Here we introduced the first rarefaction wave. We find that $u(x, t) = g_1\left(\frac{x}{t}\right) = \frac{x}{t}$.

<u>$t < x < 1 + t$:</u> Here we can use characteristics to find that $u(x, t) = u_0(x - t) = 1$.

<u>$1 + t < x < 1 + 2t$:</u> Here we introduced the second rarefaction wave. We find that $u(x, t) =$

$g_2\left(\frac{x-1}{t}\right) = \frac{x-1}{t}$.
For $x > 1+2t$ we are in the realm of characteristics. However we have two distinct domains:
$\underline{1+2t < x < 2+2t}$: Here we can use characteristics to find that $u(x,t) = u_0(x-2t) = 2$.
$\underline{x \geq 2+2t}$: Here we can use characteristics to find that $u(x,t) = u_0\left(\frac{x}{1+t}\right) = \frac{x}{1+t}$.
In conclusion:

$$u(x,t) := \begin{cases} 0, & x \leq 0, \\ \frac{x}{t}, & 0 < x \leq t, \\ 1, & t < x \leq 1+t, \\ \frac{x-1}{t}, & 1+t < x \leq 1+2t, \\ 2, & 1+2t < x \leq 2+2t, \\ \frac{x}{1+t}, & 2+2t \leq x. \end{cases}$$

d) The proposed solution is piecewise continuous, so we only need to check the continuity at the boundary of the regions. For instance:

$$\lim_{(x,t)\to(0,t)_\ell} u(x,t) = 0 = u(0,t) = \lim_{(x,t)\to(0,t)_r} u(x,t)$$

where we have used $(0,t)_\ell$ and $(0,t)_r$ to indicate we are approaching from the region to the left of the problematic curve and its right. We can similarly check all other boundaries.
e) Since our solution is continuous, and as such has no shocks, it satisfies Lax's entropy condition automatically.

Question 6 *(Q8 – May 2024 exam)*. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary. Let $F : \mathbb{R} \to \mathbb{R}$ be a given smooth function which is bounded above. We consider the energy functional

$$E[u] := \int_\Omega \frac{1}{2}(\Delta u(\vec{x}))^2 d\vec{x} - \int_\Omega F(u(\vec{x}))d\vec{x},$$

which we define on the set of scalar functions which belong to

$$\mathcal{V} := \{u \in C^2(\overline{\Omega}) : \ \nabla u \cdot \vec{n} = 0 \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Here we denoted by Δ the Laplace operator, by ∇ the gradient operator and by $\vec{n}$ the outward pointing unit normal vector field to $\partial\Omega$.

a) Show that there exists a constant $c_0 > 0$ such that $E[u] \geq -c_0$ for all $u \in \mathcal{V}$.
b) Suppose that $u \in \mathcal{V}$ is a minimiser of E. Write down the first order optimality condition, i.e. the Euler–Lagrange equation satisfied by u [*The first order optimality condition is the condition we find by using the variational method, i.e. by considering* $u_\varepsilon = u + \varepsilon\varphi$]
c) Suppose that $u \in C^4(\overline{\Omega})$ is a minimiser of E over $\mathcal{V}$. Find the PDE and boundary conditions satisfied by u.
d) Suppose that F is strictly concave. Deduce that if a minimiser of E over $\mathcal{V}$ exists, then it must be unique. [*We have not discussed this topic this year.*]
e) Show the uniqueness of minimisers of E in $\mathcal{V}$, if F is the constant zero function.

Solution:

a) From the assumptions on F we know that there exists $C > 0$ such that $F(x) \leq C$. Consequently,

$$E[u] = \int_\Omega \frac{1}{2}(\Delta u(x))^2 dx - \int_\Omega F(u(x))dx \geq -\int_\Omega C dx = -C|\Omega| = -c_0.$$

b) Assume that $u \in \mathcal{V}$ is a minimiser of E. Let $\varphi \in \mathcal{V}$ be given and consider the function

$$u_\varepsilon = u + \varepsilon\varphi.$$

Since

$$u_\varepsilon|_{\partial\Omega} = u|_{\partial\Omega} + \varepsilon\varphi|_{\partial\Omega} = 0$$

and

$$\nabla u_\varepsilon \cdot \vec{n}|_{\partial\Omega} = \nabla u \cdot \vec{n}|_{\partial\Omega} + \varepsilon\nabla\varphi \cdot \vec{n}|_{\partial\Omega} = 0$$

we conclude that $u_\varepsilon \in \mathcal{V}$ and consequently, by the assumption of minimality of u,

$$E[u_\varepsilon] \leq E[u].$$

This implies that if $E[u_\varepsilon]$ is differentiable with respect to ε then as $u_0 = u$ we must have that $\frac{d}{d\varepsilon}E[u_\varepsilon] = 0$. Since

$$E[u_\varepsilon] = \int_\Omega \frac{1}{2}(\Delta u(x) + \varepsilon\Delta\varphi(x))^2 dx - \int_\Omega F(u(x) + \varepsilon\varphi(x))dx$$

$$= \frac{1}{2}\int_\Omega (\Delta u(x))^2 dx + \varepsilon\int_\Omega \Delta u(x)\Delta\varphi(x)dx + \frac{\varepsilon^2}{2}\int_\Omega (\Delta\varphi(x))^2 - \int_\Omega F(u(x) + \varepsilon\varphi(x))dx$$

and since F is smooth, we have that

$$\frac{d}{d\varepsilon}E[u_\varepsilon] = \int_\Omega \Delta u(x)\Delta\varphi(x)dx + \varepsilon\int_\Omega (\Delta\varphi(x))^2 - \int_\Omega F'(u(x) + \varepsilon\varphi(x))\varphi(x)dx.$$

We conclude that for any $\varphi \in \mathcal{V}$ we must have

$$0 = \int_\Omega \Delta u(x)\Delta\varphi(x)dx - \int_\Omega F'(u(x))\varphi(x)dx.$$

c) To find the PDE that u must satisfy we need to rewrite our optimality condition in a way that involves only φ (and not its derivatives). This means that we will need to integrate by parts, which is justified by the assumption that $u \in C^4\left(\overline{\Omega}\right)$. We have that

$$\begin{aligned}\Delta u\Delta\varphi &= \Delta u \operatorname{div}\left(\nabla\varphi\right) = \operatorname{div}\left(\Delta u\nabla\varphi\right) - \nabla\varphi\nabla\Delta u \\ &= \operatorname{div}\left(\Delta u\nabla\varphi\right) - \left(\operatorname{div}\left(\varphi\nabla\Delta u\right) - \varphi\operatorname{div}(\nabla\Delta u)\right) \\ &= \operatorname{div}\left(\Delta u\nabla\varphi\right) - \operatorname{div}\left(\varphi\nabla\Delta u\right) + \varphi\Delta^2 u.\end{aligned}$$

Using the divergence theorem we find that

$$0 = \int_\Omega \Delta u(x)\Delta\varphi(x)dx - \int_\Omega F'(u(x))\varphi(x)dx = \int_{\partial\Omega} \Delta u(y)\nabla\varphi(y)\cdot \vec{n(y)}dS(y)$$

$$- \int_{\partial\Omega} \varphi(y)\nabla\Delta u(y)\cdot \vec{n(y)}dS(y) + \int_\Omega \Delta^2 u(x)\varphi(x)dx - \int_\Omega F'(u(x))\varphi(x)dx$$

$$= \int_\Omega \left(\Delta^2 u(x) - F'(u(x))\right)\varphi(x)dx,$$

where we have used the fact that $\varphi \in \mathcal{V}$. Choosing $\varphi \in C_c^1(\Omega) \subset \mathcal{V}$ and using the fundamental lemma of the calculus of variation we conclude the u must satisfy

$$\Delta^2 u(x) = F'(u(x)), \qquad x \in \Omega.$$

As $u \in \mathcal{V}$ we have the boundary conditions $u|_{\partial\Omega} = 0$, $\nabla u \cdot \vec{n}|_{\partial\Omega} = 0$.

d) We have not discussed this topic this year.

e) Assume that we have two minimisers u_1 and u_2. Since $F = 0$ we must have that for any $\varphi \in \mathcal{V}$

$$\int_{\Omega} \Delta u_1(x) \Delta\varphi(x)\, dx = 0 = \int_{\Omega} \Delta u_2(x) \Delta\varphi(x)\, dx.$$

In particular

$$\int_{\Omega} \Delta \left(u_1(x) - u_2(x)\right) \Delta\varphi(x)\, dx = 0.$$

Choosing $\varphi = u_1 - u_2 \in \mathcal{V}$ we find that

$$\int_{\Omega} \left(\Delta \left(u_1(x) - u_2(x)\right)\right)^2 dx = 0$$

which implies

$$\Delta \left(u_1(x) - u_2(x)\right) = 0, \qquad x \in \Omega.$$

This implies that $u_1 - u_2$ is harmonic and is zero on the boundary (even if the set is not connected). Consequently, $u_1 \equiv u_2$.