PARTIAL DIFFERENTIAL EQUATIONS III & V
REVISION CLASS

Question 1 *(Q2 – May 2024 exam).* We consider the following Cauchy problem for the scalar unknown function u that we aim to solve by the method of characteristics.

$$(1)\qquad \begin{cases} 5 - x_1^2 \partial_{x_2} u(x_1, x_2) = 0, & (x_1, x_2) \in \mathbb{R}^2, \\ u(x_1, 0) = 3, & x_1 \in \mathbb{R}. \end{cases}$$

a) Determine the leading vector field, the Cauchy datum and the Cauchy curve associated to this problem.
b) Find all the points on the Cauchy curve which are noncharacteristic.
c) Write down the ODE system for the characteristics and for the solution along the characteristics. Then solve this system.
d) Sketch a few characteristic curves.
e) Find the solution u to (1). Determine its maximal domain of definition.

Sol:

(a) The eq. reads as

$$\begin{cases} x_1^2 \partial_{x_2} u(x_1, x_2) = 5 & (x_1, x_2) \in \mathbb{R}^2 \\ u(x_1, 0) = 3 & x_1 \in \mathbb{R} \end{cases}$$

This implies that the leading vector field is

$$\vec{a}(x_1, x_2) = (0, x_1^2)$$

The Cauchy curve is

$$\gamma(s) = (s, 0) \qquad s \in \mathbb{R} \quad (x_1\text{-axis})$$

The Cauchy datum is

$$u(\gamma(s)) = u(s, 0) = 3$$

(b) $\gamma'(s) = (1, 0)$. $(s, 0)$ is non-char. if

$$\vec{a}(\gamma(s)) \cdot (0, -1) \neq 0$$

$0 \neq \vec{a}(\gamma(s)) \cdot (0,-1) = (0, s^2) \cdot (0,-1) = -s^2$

i.e $s \neq 0$.

(c)
$$\partial_\tau x_1(s,\tau) = 0$$
$$\partial_\tau x_2(s,\tau) = x_1^2(s,\tau)$$
$$\partial_\tau z(s,\tau) = 5$$

and

$x_1(0,s) = s$, $x_2(0,s) = 0$, $z(0,s) = u(\gamma(s)) = 3$

we find that

$x_1(\tau,s) = x_1(0,s) = s$

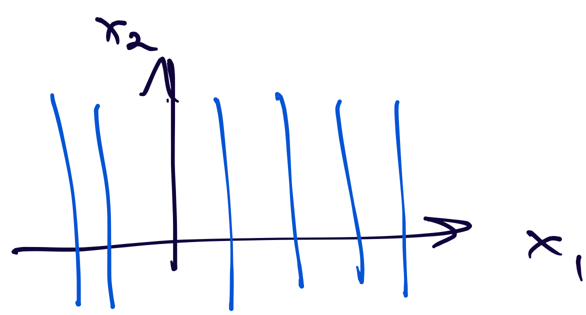
$z(\tau,s) = 5\tau + z(0,s) = 5\tau + 3$

Also

$\partial_\tau x_2(\tau,s) = x_1^2(\tau,s) = s^2$

$\Rightarrow \quad x_2(\tau,s) = s^2\tau + x_2(0,s) = s^2\tau$

(d)



(e) invert the map $\tau, s \longrightarrow x_1, x_2$ and then $u(x_1,x_2) = z(\tau(x_1,x_2), s(x_1,x_2))$.

We have

$$x_1 = s$$

$$x_2 = s^2 \tau = x_1^2 \tau$$

$$\Rightarrow \quad \tau = \frac{x_2}{x_1^2} \quad \left(\begin{array}{l}\text{recall } s \neq 0 \\ \text{i.e. } x_1 \neq 0\end{array}\right)$$

$$u(x_1, x_2) = z\left(\frac{x_2}{x_1^2}, x_1\right) = 5\frac{x_2}{x_1^2} + 3$$

max. domain of def is

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq 0\}$$

Question 2 *(Q3 – May 2024 exam)*. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary, and suppose that $d \geq 2$.

a) Suppose that $u, v \in C^2(\overline{\Omega})$. Show that the following formula holds
$$\int_\Omega v \Delta u d\vec{x} + \int_\Omega \nabla v \cdot \nabla u d\vec{x} = \int_{\partial\Omega} v \partial_n u dS,$$
where ∇ stands for the gradient, Δ stands for the Laplace operator and we used the notation $\partial_n u = \nabla u \cdot \vec{n}$, with $\vec{n}$ being the outward pointing unit normal vector to $\partial\Omega$.

b) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that
$$\int_{\partial\Omega} \partial_n u dS = 0.$$

c) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that $\int_{\partial\Omega} u \partial_n u dS$ is nonnegative.

d) Suppose that $u, v : \Omega \to \mathbb{R}$ are both harmonic and $u, v \in C^2(\overline{\Omega})$. Show that
$$\int_{\partial\Omega} (u \partial_n v - v \partial_n u)\, dS = 0.$$

Sol: (a) for $u, v \in C^2(\overline{\Omega})$

$$v \Delta u = v \operatorname{div}(\nabla u) = \operatorname{div}(v \nabla u) - \nabla v \cdot \nabla u$$

integrating and using the divergence theorem we find that

$$\int_\Omega v(x) \Delta u(x) dx = \int_{\partial\Omega} v(y) \underbrace{\nabla u(y) \cdot n(y)}_{\partial_n u(y)} dS(y) - \int_\Omega \nabla v(x) \cdot \nabla u(x) dx$$

which is the desired result.

(b) choosing $v \equiv 1$ and using our formula we get

$$\int_\Omega 1 \cdot \underbrace{\cancel{\Delta u(x)}}_{0 \text{ since } u \text{ is harmonic}} dx = \int_{\partial\Omega} \partial_n u(y) dS(y) - \int_\Omega \underbrace{\cancel{0 \cdot \nabla u(x)}}_{0} dx$$

$$\Rightarrow \int_{\partial\Omega} \partial_n u(y) dS(y) = 0$$

(c) choosing $v=u$ we get

$$\int_\Omega u(x)\underbrace{\cancel{\Delta u(x)}}_{\substack{0 \\ u \text{ is harmonic}}}\,dx = \int_{\partial\Omega} u(y)\partial_n u(y)\,dS(y) - \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$

$$\Rightarrow \int_{\partial\Omega} u(y)\partial_n u(y)\,dS(y) = \int_\Omega |\nabla u(x)|^2\,dx \geq 0$$

(d) We have that

$$\int_\Omega v(x)\underbrace{\cancel{\Delta u(x)}}_{\substack{0 \\ u \text{ is harmonic}}}\,dx + \int_\Omega \nabla v(x)\cdot\nabla u(x)\,dx = \int_{\partial\Omega} v(y)\partial_n u(y)\,dS(y)$$

and

$$\int_\Omega u(x)\underbrace{\cancel{\Delta v(x)}}_{\substack{0 \\ v \text{ is harmonic}}}\,dx + \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = \int_{\partial\Omega} u(y)\partial_n v(y)\,dS(y)$$

subtracting these identities gives

$$0 = \int_{\partial\Omega} \left[v(y)\partial_n u(y) - u(y)\partial_n v(y)\right] dS(y).$$

Question 3 *(Q5 – May 2024 exam).* Let $\alpha \in \mathbb{R}$ and set $A^\alpha = (a^\alpha_{ij})^2_{i,j=1} \in \mathbb{R}^{2\times 2}$ to be the matrix $A^\alpha := \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$. For a given open set $\Omega \subseteq \mathbb{R}^2$ and $u \in C^2(\Omega)$, we define the differential operator

$$(\mathcal{L}^\alpha u)(\vec{x}) := -A^\alpha : D^2 u(\vec{x}) = -\sum_{i,j=1}^{2} a^\alpha_{ij} \partial_{x_i} \partial_{x_j} u(\vec{x}),$$

where $D^2 u$ stands for the Hessian matrix of u.

a) Show that the matrix A^α is positive semi-definite if and only if $|\alpha| \le 1$. Show that A^α is positive definite if and only if $|\alpha| < 1$.
b) Let Ω be open, bounded and connected with smooth boundary. Suppose that $|\alpha| < 1$ and $u : \Omega \to \mathbb{R}$ is a classical solution to

$$(\mathcal{L}^\alpha u)(\vec{x}) = 0, \ \ \vec{x} \in \Omega.$$

Explain why u attains both its minimum and maximum on $\partial\Omega$.
c) Now we set $\alpha = 1$. Find all those real numbers $c_1, c_2 \in \mathbb{R}$ for which the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$u(x_1, x_2) = c_1(x_1^2 + x_2^2) - c_2 x_1 x_2$$

is a solution to $\mathcal{L}^1 u = 0$.
d) Suppose that we are in the setting of the previous point (c). Show that u fails to satisfy either the strong minimum or the strong maximum principle (one of the two). [*Hint*: choose c_1, c_2 such that $u(x_1, x_2) \ge 0$ for all $(x_1, x_2) \in \mathbb{R}^2$. Find a particular bounded connected domain $\Omega \subset \mathbb{R}^2$, which is a sublevel set of u, i.e. $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : \ u(x_1, x_2) < r\}$, for some $r > 0$. Deduce the failure of the strong minimum principle in this domain.
Remark: This is a corrected version of the problem which asked about the weak minimum/maximum principle.]

Sol:

(a) A symmetric matrix A is positive semi-def if and only if its eigenvalues λ_1, λ_2 are non-negative, and is positive def. if $\lambda_1, \lambda_2 > 0$.

Recall that

$\det A = \lambda_1 \cdot \lambda_2 \qquad tr(A) = \lambda_1 + \lambda_2$

A is positive def $\Leftrightarrow \det A > 0, \ tr(A) > 0$

A is positive semi def $\Leftrightarrow \det A \ge 0, \ tr(A) \ge 0$

In our case

$\det A^\alpha = 1 - \alpha^2 \qquad tr(A^\alpha) = 2$

$\Rightarrow$ A^α is pos. def $\Leftrightarrow$ $1-\alpha^2>0$

A^α is pos. semi def. $\Leftrightarrow$ $1-\alpha^2 \geq 0$.

(b) Since $|\alpha|<1$, A^α is positive def. and as such L^α is an elliptic operator. Consequently any u that solves $L^\alpha u=0$ will satisfy the weak min and max principles.

(Careful $u \in C^2(\Omega) \cap C(\overline{\Omega})$).

(c) $$L^1 u = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : D^2 u$$

$$= \left(-\partial^2_{x_1 x_1} - \partial^2_{x_2 x_2} - 2\partial^2_{x_1 x_2}\right) u$$

$$u(x_1,x_2) = c_1(x_1^2+x_2^2) + c_2 x_1 x_2$$

$$\partial^2_{x_1 x_1} u = 2c_1 = \partial^2_{x_2 x_2} u$$

$$\partial^2_{x_1 x_2} u = c_2$$

$$L^1 u(x_1,x_2)=0 \Leftrightarrow 0 = -2c_1 - 2c_1 - 2c_2$$

$$\Rightarrow \quad c_2 = -2c_1$$

Note: $u(x_1,x_2) = c_1(x_1^2+x_2^2) - 2c_1 x_1 x_2$

$$= c_1 (x_1 - x_2)^2$$

(d) for $c_1 > 0$ we find that

$u(x_1, x_2) \geqslant 0 \qquad u \not\equiv 0$

$u\big|_{x_1 = x_2} = 0$. Any domain Ω that contains a point from the line $x_1 = x_2$ will have a local minimum as $u \geqslant 0$. However $u \not\equiv 0$.

Question 4 *(Q6 – May 2024 exam)*. Let $f : \mathbb{R} \to \mathbb{R}$ of class C^2 be given. Suppose that this is strongly convex, i.e. there exists $c_0 > 0$ such that $f''(x) \geq c_0$ for all $x \in \mathbb{R}$. Consider the following Cauchy problem for the unknown $u : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$

$$\begin{cases} \partial_t u(x,t) + \partial_x(f(u(x,t))) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

For $\varepsilon > 0$ we consider the following approximation of (2)

$$\begin{cases} \partial_t u^\varepsilon(x,t) + \partial_x(f(u^\varepsilon(x,t))) - \varepsilon \partial^2_{xx} u^\varepsilon(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u^\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

a) State Lax's entropy condition for weak solutions to the Cauchy problem (2).

b) We look for a solution to (3) in the form

$$u^\varepsilon(x,t) := v\left(\frac{x - \alpha t}{\varepsilon}\right), \tag{4}$$

for a given constant $\alpha \in \mathbb{R}$ and some given smooth enough function $v : \mathbb{R} \to \mathbb{R}$. Find the second order ODE that v needs to satisfy in order for the formula (4) to give a classical solution to (3).

c) Let $u_\ell, u_r \in \mathbb{R}$ be given, and we are looking for a solution to the ODE for v found in (b) with the additional assumptions

$$\lim_{s\to-\infty} v(s) = u_\ell; \quad \lim_{s\to+\infty} v(s) = u_r; \quad \lim_{s\to\pm\infty} v'(s) = 0.$$

Suppose that we find such a solution v. Compute the limit $\lim_{\varepsilon\to 0} u^\varepsilon(x,t)$, in the case when $x \neq \alpha t$.

d) Suppose that we are in the setting of (c). Find an equation that α needs to satisfy, in terms of f and u_ℓ, u_r. [*Hint*: integrate the second oder ODE for v, then take limits $s \to \pm\infty$].

e) Suppose that $u_0(x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases}$ Suppose that $u_r < u_l$. Suppose that (3) has a classical solution in the form of (4), and v and α satisfy all the previously set and obtained properties. Conclude that $u^\varepsilon(x,t) \to u(x,t)$, as $\varepsilon \to 0$, almost everywhere, where u is the unique solution to (2) which satisfies Lax's entropy condition.

Sol:
(a) Lax entropy cond. states that for any shock curve of the sol. $(\sigma(t), t)$ we have $f'(u_r) < \dot{\sigma}(t) < f'(u_l)$

b)
$$\partial_t u^\varepsilon(x,t) = -\frac{\alpha}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right)$$
$$\partial_x u^\varepsilon(x,t) = \frac{1}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right)$$
$$\partial^2_{xx} u^\varepsilon(x,t) = \frac{1}{\varepsilon^2} v''\left(\frac{x-\alpha t}{\varepsilon}\right)$$

Note: $\partial_x(f(u^\varepsilon(x,t))) = f'(u^\varepsilon(x,t)) \cdot \partial_x u^\varepsilon(x,t)$

Our PDE becomes:

$$-\frac{\alpha}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right) + f'\left(v\left(\frac{x-\alpha t}{\varepsilon}\right)\right)\cdot \frac{1}{\varepsilon} v'\left(\frac{x-\alpha t}{\varepsilon}\right)$$

$$-\varepsilon \cdot \frac{1}{\varepsilon^2} v''\left(\frac{x-\alpha t}{\varepsilon}\right) = 0$$

Replace $\frac{x-\alpha t}{\varepsilon}$ by s we get (ε cancelled)

$$-\alpha v'(s) + \underbrace{f'(v(s))v'(s)}_{\frac{d}{ds} f(v(s))} - v''(s) = 0$$

(c)

$$u^{\varepsilon}(x,t) = v\left(\frac{x-\alpha t}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0^+]{} \begin{cases} u_r & x > \alpha t \\ u_l & x < \alpha t \end{cases}$$

u_r: $v(\infty)$, u_l: $v(-\infty)$

(d) $$v''(s) = -\alpha v'(s) + \frac{d}{ds}\left(f(v(s))\right)$$

Integrating we get that

$$v'(s) = -\alpha v(s) + f(v(s)) + C$$

Taking $s \to \infty$ we get

$$0 = -\alpha u_r + f(u_r) + C$$

Taking $s \to -\infty$ we get

$$0 = -\alpha u_l + f(u_l) + C$$

subtracting we find that

$$\alpha(u_r - u_l) = f(u_r) - f(u_l)$$

i.e

$$\alpha = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

e) we have

$$u(x,t) = \begin{cases} u_r & x > \alpha t \\ u_l & x < \alpha t \end{cases}$$ defined a.e.

$u(x,0) = u_0(x)$.

On $x > \alpha t$, $x < \alpha t$ $u(x,t)$ solves

$$\underbrace{\partial_t u}_{0} + f'(u(x,t))\underbrace{\partial_x u}_{0} = 0$$

if we show ~~that~~ u satisfies Lax's condition we will conclude the proof.

The shock curve we have is

$$s(t) = \alpha t$$

$$\dot{s}(t) = \alpha = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

Recall $\exists\, c \in (u_r, u_l)$ (since $u_r < u_l$)

$$\frac{f(u_r) - f(u_l)}{u_r - u_l} = f'(c)$$

f is strictly convex $\to f' \nearrow$ $\Rightarrow f'(u_r) < f'(c) < f'(u_l)$
we get the concl.

Question 5 *(Q7 – May 2024 exam)*. We consider the following Cauchy problem

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

We set

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 < x < 1, \\ 2, & 1 < x < 2, \\ x, & 2 < x. \end{cases}$$

We aim to construct a unique entropy solution to this Cauchy problem.

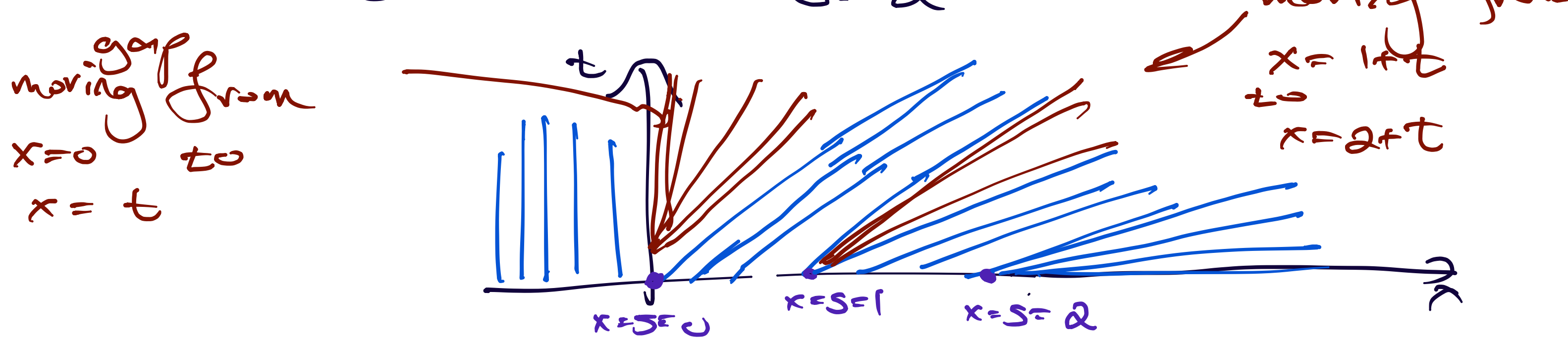
a) Sketch the characteristic lines associated with the Cauchy problem and discuss about the need of shock curves and/or rarefaction waves.
b) Introduce the corresponding shocks and/or rarefaction waves.
c) Write down the candidate for the weak entropy solutions to (5).
d) Show that this solution is continuous everywhere if $t > 0$.
e) Show that the solution satisfies Lax's entropy condition.

Sol:

(a) on any interval where u_0 is cont. the method of char. As the Cauchy curve is $(s,0)$ and $c(u) = u$ we find that the char. are lines of the form $x = s + u_0(s)t \qquad s \in \mathbb{R}$

i.e.

$$x = \begin{cases} s & s < 0 \\ s+t & 0 < s < 1 \\ s+2t & 1 < s < 2 \\ s+st & s > 2 \end{cases}$$



we need two rarefcation waves.

(b) We have two domains of rarefcations

$0<x<t$: Here we guess a sol. of the form $g_1(x/t)$ with

$$g_1(0)=0, \quad g_1(1)=1$$

$1+t<x<1+2t$: Here we guess a sol of the form $g_2\left(\frac{x-1}{t}\right)$ with

$$g_2(1)=1, \quad g_2(2)=2$$

Guessing a sol. of the form $h\left(\frac{x-x_0}{t}\right)$ to

$$\partial_t u + u\partial_x u = 0$$

gives us

$$-\frac{x-x_0}{t^2}h'\left(\frac{x-x_0}{t}\right) + \frac{h\left(\frac{x-x_0}{t}\right)}{t}h'\left(\frac{x-x_0}{t}\right) = 0$$

which reduces to the ODE

$$h'(z)\left(h(z)-z\right)=0$$

Noticing that $h(z)=z$ solves this ODE we conclude that a rarefaction sol. with our initial cond. is

$$g_1\left(\frac{x}{t}\right)=x/t \qquad g_2\left(\frac{x-1}{t}\right)=\frac{x-1}{t}.$$

(c) The candidate for the sol. is:

$$u(x,t) = \begin{cases} 0 & x<0 \\ x/t & 0<x<t \\ 1 & t<x<1+t \\ \frac{x-1}{t} & 1+t<x<1+2t \\ 2 & 1+2t<x<2+2t \\ \frac{x}{1+t} & x>2+2t \end{cases}$$

char. $x=s$ $u_0(s)=0$

rarefaction

char. $s=x-t$ $u_0(s)=1$

rarefaction

char. $s=x-2t$ $u_0(s)=2$

char. $x=s+st$ $\Rightarrow s=\frac{x}{1+t}$ $u_0(s)=s$

(d) Cont. follows from checking the curves we stitch. For example

The curve $x<0$

$$\lim_{(x,t)\to \text{left of the curve}} u(x,t) = \lim_{(x,t)\to \text{left} \dots} 0 = 0$$

$$\lim_{(x,t)\to \text{right of the curve}} u(x,t) = \lim_{(x,t)\to \text{right} \dots} x/t = 0 \quad \left(\text{as } x\to 0 \text{ and } t>0\right)$$

(e) As u is cont. there are no shocks and Lax's entropy cond. is satisfied.

Question 6 *(Q8 – May 2024 exam).* Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary. Let $F : \mathbb{R} \to \mathbb{R}$ be a given smooth function which is bounded above. We consider the energy functional

$$E[u] := \int_\Omega \frac{1}{2}(\Delta u(\vec{x}))^2 d\vec{x} - \int_\Omega F(u(\vec{x}))d\vec{x},$$

which we define on the set of scalar functions which belong to

$$\mathcal{V} := \{u \in C^2(\overline{\Omega}) : \ \nabla u \cdot \vec{n} = 0 \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Here we denoted by Δ the Laplace operator, by ∇ the gradient operator and by $\vec{n}$ the outward pointing unit normal vector field to $\partial\Omega$.

a) Show that there exists a constant $c_0 > 0$ such that $E[u] \geq -c_0$ for all $u \in \mathcal{V}$.
b) Suppose that $u \in \mathcal{V}$ is a minimiser of E. Write down the first order optimality condition, i.e. the Euler–Lagrange equation satisfied by u [*The first order optimality condition is the condition we find by using the variational method, i.e. by considering $u_\varepsilon = u + \varepsilon\varphi$*]
c) Suppose that $u \in C^4(\overline{\Omega})$ is a minimiser of E over $\mathcal{V}$. Find the PDE and boundary conditions satisfied by u.
d) Suppose that F is strictly concave. Deduce that if a minimiser of E over $\mathcal{V}$ exists, then it must be unique. [*We have not discussed this topic this year.*]
e) Show the uniqueness of minimisers of E in $\mathcal{V}$, if F is the constant zero function.

Sol: (a) $\exists\, C>0$ s.t.

$$F(z) \leq C \quad \forall z \in \mathbb{R}$$

$$\Rightarrow E[u] = \frac{1}{2}\int_\Omega (\Delta u)^2 dx - \int_\Omega F(u(x))\,dx$$

$$\geq -\int_\Omega F(u(x))\,dx \geq -\int_\Omega C\,dx$$

$$= -C|\Omega| = -c_0$$

(b) Given $\varphi \in \mathcal{V}$ we have that for any $u \in \mathcal{V}$ and $\varepsilon > 0$

$$u_\varepsilon = u + \varepsilon\varphi \in C^2(\overline{\Omega})$$

$$u_\varepsilon|_{\partial\Omega} = u|_{\partial\Omega} + \varepsilon\varphi|_{\partial\Omega} = 0 \qquad u, \varphi \in \mathcal{V}$$

$$\partial_n u_\varepsilon|_{\partial\Omega} = \partial_n u|_{\partial\Omega} + \varepsilon\partial_n\varphi|_{\partial\Omega} = 0 \qquad u, \varphi \in \mathcal{V}$$

If u is a minimiser then

$$E[u_\varepsilon] \geq E[u] \quad \forall \varepsilon \in \mathbb{R}$$

$u_{\varepsilon=0} = u \Rightarrow$ if $E[u_\varepsilon]$ is diff w.r.t ε then

$$\frac{d}{d\varepsilon} E[u_\varepsilon]\Big|_{\varepsilon=0} = 0$$

$$E[u_\varepsilon] = \frac{1}{2}\int_\Omega (\Delta u(x) + \varepsilon \Delta \varphi(x))^2 dx - \int_\Omega F(u(x) + \varepsilon \varphi(x)) dx$$

$u, \varphi \in C^2(\overline{\Omega})$, F is smooth so we can diff. under the integral

$$\frac{d}{d\varepsilon} E[u_\varepsilon] = \int_\Omega (\Delta u(x) + \varepsilon \Delta\varphi(x)) \Delta\varphi(x)\, dx - \int_\Omega F'(u(x) + \varepsilon\varphi(x)) \cdot \varphi(x)\, dx$$

We get

$$0 = \frac{d}{d\varepsilon} E[u_\varepsilon]\Big|_{\varepsilon=0} = \int_\Omega \Delta u(x) \Delta\varphi(x)\, dx - \int_\Omega F'(u(x))\varphi(x)\, dx$$

(c) $\Delta u \Delta \varphi = \Delta u \, \mathrm{div}(\nabla \varphi) = \mathrm{div}(\Delta u \nabla \varphi) - \nabla\varphi \nabla(\Delta u) = \mathrm{div}(\Delta u \nabla \varphi)$

$$-\left[\mathrm{div}(e\nabla\Delta u) - e\,\mathrm{div}(\nabla\Delta u)\right]$$

$$= \mathrm{div}(\Delta u\nabla e) - \mathrm{div}(e\nabla(\Delta u)) + e\,\mathrm{div}(\nabla\Delta u)$$

Using the divergence theorem we find that

since $\partial_n e = 0$ on $\partial\Omega$ ($e\in V$)

$$\int_\Omega \Delta u(x)\Delta e(x)\,dx = \int_{\partial\Omega}\Delta u(y)\,\underbrace{\cancel{\nabla e(y)\cdot\hat{n}(y)}}_{0}\,dS(y)$$

$$-\int_{\partial\Omega}\underbrace{\cancel{e(y)}}_{0}\,\nabla(\Delta u)(y)\cdot\hat{n}(y)\,dS(y) + \int_\Omega e(x)\Delta^2 u(x)\,dx$$

since $e|_{\partial\Omega}=0$ ($e\in V$)

Using our optimality eq. we conclude that when $u\in C^4(\overline{\Omega})$ is a minimiser

$$0 = \int_\Omega\left[\Delta^2 u(x) - F'(u(x))\right]e(x)\,dx$$

This holds for all $e\in V$. Noticing that $C^1_c(\Omega)\subset V$ we use the fundamental lemma of Calculus of Variation to conclude that

$$\Delta^2 u(x) = F'(u(x)) \qquad x\in\Omega$$

The boundary conditions are given by

the fact that $u \in V$ i.e

$$u\big|_{\partial\Omega} = 0 \qquad \partial_n u\big|_{\partial\Omega} = 0.$$

(d) We didn't discuss this topic.

(e) Assuming $F \equiv 0$ and that $u_1, u_2 \in V$ are minimisers we find that $\forall e \in V$

$$0 = \int_\Omega \Delta u_1(x) \Delta e(x)\, dx = \int_\Omega \Delta u_2(x) \Delta e(x)\, dx$$

$\Rightarrow$

$$0 = \int_\Omega \Delta(u_1 - u_2)(x)\, \Delta e(x)\, dx$$

as $u_1, u_2 \in V$ so is $e = u_1 - u_2$ and consequently

$$0 = \int \left(\Delta(u_1 - u_2)(x)\right)^2 dx$$

$\Rightarrow$ $(u_1, u_2 \in C^2(\overline{\Omega}))$

$$\Delta(u_1 - u_2) = 0$$

i.e. $u_1 - u_2$ is harmonic. As $u_1 - u_2\big|_{\partial\Omega} = 0$ we can use the max/min principles to conclude that $u_1 - u_2 \equiv 0$ i.e.

$$u_1 = u_2$$