PARTIAL DIFFERENTIAL EQUATIONS III & V
REVISION CLASS

Question 1 *(Q2 – May 2024 exam).* We consider the following Cauchy problem for the scalar unknown function u that we aim to solve by the method of characteristics.

$$\begin{cases} 5 - x_1^2 \partial_{x_2} u(x_1, x_2) = 0, & (x_1, x_2) \in \mathbb{R}^2, \\ u(x_1, 0) = 3, & x_1 \in \mathbb{R}. \end{cases} \tag{1}$$

a) Determine the leading vector field, the Cauchy datum and the Cauchy curve associated to this problem.
b) Find all the points on the Cauchy curve which are noncharacteristic.
c) Write down the ODE system for the characteristics and for the solution along the characteristics. Then solve this system.
d) Sketch a few characteristic curves.
e) Find the solution u to (1). Determine its maximal domain of definition.

Question 2 *(Q3 – May 2024 exam).* Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary, and suppose that $d \geq 2$.

a) Suppose that $u, v \in C^2(\overline{\Omega})$. Show that the following formula holds

$$\int_\Omega v \Delta u \, d\vec{x} + \int_\Omega \nabla v \cdot \nabla u \, d\vec{x} = \int_{\partial\Omega} v \partial_n u \, dS,$$

where ∇ stands for the gradient, Δ stands for the Laplace operator and we used the notation $\partial_n u = \nabla u \cdot \vec{n}$, with $\vec{n}$ being the outward pointing unit normal vector to $\partial\Omega$.
b) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that

$$\int_{\partial\Omega} \partial_n u \, dS = 0.$$

c) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and $u \in C^2(\overline{\Omega})$. Show that $\int_{\partial\Omega} u \partial_n u \, dS$ is nonnegative.
d) Suppose that $u, v : \Omega \to \mathbb{R}$ are both harmonic and $u, v \in C^2(\overline{\Omega})$. Show that

$$\int_{\partial\Omega} (u \partial_n v - v \partial_n u) \, dS = 0.$$

Question 3 *(Q5 – May 2024 exam).* Let $\alpha \in \mathbb{R}$ and set $A^\alpha = (a^\alpha_{ij})^2_{i,j=1} \in \mathbb{R}^{2\times 2}$ to be the matrix $A^\alpha := \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$. For a given open set $\Omega \subseteq \mathbb{R}^2$ and $u \in C^2(\Omega)$, we define the differential operator

$$(\mathcal{L}^\alpha u)(\vec{x}) := -A^\alpha : D^2 u(\vec{x}) = -\sum_{i,j=1}^{2} a^\alpha_{ij} \partial_{x_i} \partial_{x_j} u(\vec{x}),$$

where $D^2 u$ stands for the Hessian matrix of u.

a) Show that the matrix A^α is positive semi-definite if and only if $|\alpha| \leq 1$. Show that A^α is positive definite if and only if $|\alpha| < 1$.

b) Let Ω be open, bounded and connected with smooth boundary. Suppose that $|\alpha| < 1$ and $u : \Omega \to \mathbb{R}$ is a classical solution to

$$(\mathcal{L}^\alpha u)(\vec{x}) = 0, \quad \vec{x} \in \Omega.$$

Explain why u attains both its minimum and maximum on $\partial\Omega$.

c) Now we set $\alpha = 1$. Find all those real numbers $c_1, c_2 \in \mathbb{R}$ for which the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$u(x_1, x_2) = c_1(x_1^2 + x_2^2) - c_2 x_1 x_2$$

is a solution to $\mathcal{L}^1 u = 0$.

d) Suppose that we are in the setting of the previous point (c). Show that u fails to satisfy either the strong minimum or the strong maximum principle (one of the two). [*Hint*: choose c_1, c_2 such that $u(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$. Find a particular bounded connected domain $\Omega \subset \mathbb{R}^2$, which is a sublevel set of u, i.e. $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : \ u(x_1, x_2) < r\}$, for some $r > 0$. Deduce the failure of the strong minimum principle in this domain.
Remark: This is a corrected version of the problem which asked about the weak minimum/maximum principle.]

Question 4 *(Q6 – May 2024 exam)*. Let $f : \mathbb{R} \to \mathbb{R}$ of class C^2 be given. Suppose that this is strongly convex, i.e. there exists $c_0 > 0$ such that $f''(x) \geq c_0$ for all $x \in \mathbb{R}$. Consider the following Cauchy problem for the unknown $u : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$

$$\begin{cases} \partial_t u(x,t) + \partial_x(f(u(x,t))) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

For $\varepsilon > 0$ we consider the following approximation of (2)

$$\begin{cases} \partial_t u^\varepsilon(x,t) + \partial_x(f(u^\varepsilon(x,t))) - \varepsilon \partial_{xx}^2 u^\varepsilon(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u^\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

a) State Lax's entropy condition for weak solutions to the Cauchy problem (2).

b) We look for a solution to (3) in the form

$$u^\varepsilon(x,t) := v\left(\frac{x - \alpha t}{\varepsilon}\right), \tag{4}$$

for a given constant $\alpha \in \mathbb{R}$ and some given smooth enough function $v : \mathbb{R} \to \mathbb{R}$. Find the second order ODE that v needs to satisfy in order for the formula (4) to give a classical solution to (3).

c) Let $u_\ell, u_r \in \mathbb{R}$ be given, and we are looking for a solution to the ODE for v found in (b) with the additional assumptions

$$\lim_{s \to -\infty} v(s) = u_\ell; \quad \lim_{s \to +\infty} v(s) = u_r; \quad \lim_{s \to \pm\infty} v'(s) = 0.$$

Suppose that we find such a solution v. Compute the limit $\lim\limits_{\varepsilon \to 0} u^\varepsilon(x,t)$, in the case when $x \neq \alpha t$.

d) Suppose that we are in the setting of (c). Find an equation that α needs to satisfy, in terms of f and u_ℓ, u_r. [*Hint*: integrate the second oder ODE for v, then take limits $s \to \pm\infty$].

e) Suppose that $u_0(x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases}$ Suppose that $u_r < u_l$. Suppose that (3) has a classical solution in the form of (4), and v and α satisfy all the previously set and obtained properties. Conclude that $u^\varepsilon(x,t) \to u(x,t)$, as $\varepsilon \to 0$, almost everywhere, where u is the unique solution to (2) which satisfies Lax's entropy condition.

Question 5 *(Q7 – May 2024 exam)*. We consider the following Cauchy problem

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R}\times(0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

We set

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 < x < 1, \\ 2, & 1 < x < 2, \\ x, & 2 < x. \end{cases}$$

We aim to construct a unique entropy solution to this Cauchy problem.

a) Sketch the characteristic lines associated with the Cauchy problem and discuss about the need of shock curves and/or rarefaction waves.
b) Introduce the corresponding shocks and/or rarefaction waves.
c) Write down the candidate for the weak entropy solutions to (5).
d) Show that this solution is continuous everywhere if $t > 0$.
e) Show that the solution satisfies Lax's entropy condition.

Question 6 *(Q8 – May 2024 exam)*. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with smooth boundary. Let $F : \mathbb{R} \to \mathbb{R}$ be a given smooth function which is bounded above. We consider the energy functional

$$E[u] := \int_\Omega \frac{1}{2}(\Delta u(\vec{x}))^2 d\vec{x} - \int_\Omega F(u(\vec{x}))d\vec{x},$$

which we define on the set of scalar functions which belong to

$$\mathcal{V} := \{u \in C^2(\overline{\Omega}) : \ \nabla u \cdot \vec{n} = 0 \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Here we denoted by Δ the Laplace operator, by ∇ the gradient operator and by $\vec{n}$ the outward pointing unit normal vector field to $\partial\Omega$.

a) Show that there exists a constant $c_0 > 0$ such that $E[u] \geq -c_0$ for all $u \in \mathcal{V}$.
b) Suppose that $u \in \mathcal{V}$ is a minimiser of E. Write down the first order optimality condition, i.e. the Euler–Lagrange equation satisfied by u [*The first order optimality condition is the condition we find by using the variational method, i.e. by considering $u_\varepsilon = u + \varepsilon\varphi$*]
c) Suppose that $u \in C^4(\overline{\Omega})$ is a minimiser of E over $\mathcal{V}$. Find the PDE and boundary conditions satisfied by u.
d) Suppose that F is strictly concave. Deduce that if a minimiser of E over $\mathcal{V}$ exists, then it must be unique. [*We have not discussed this topic this year.*]
e) Show the uniqueness of minimisers of E in $\mathcal{V}$, if F is the constant zero function.