

Partial Differential Equations III & V, Exercise Sheet 6: Solutions

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1. The Fourier transform: The heat equation with source term.

(i) By the Fundamental Theorem of Calculus

$$\begin{aligned}\dot{x}(t) &= \lambda G e^{\lambda t} + e^{\lambda(t-s)} F(s) \Big|_{s=t} + \int_0^t \lambda e^{\lambda(t-s)} F(s) ds \\ &= \lambda G e^{\lambda t} + F(t) + \lambda \int_0^t e^{\lambda(t-s)} F(s) ds \\ &= \lambda x(t) + F(t)\end{aligned}$$

as claimed.

(ii) Taking the Fourier transform of $u_t = k u_{xx} + f$ with respect to the x variable gives

$$\widehat{u}_t = k \widehat{u_{xx}} + \widehat{f} \iff \hat{u}_t(\xi, t) = k(i\xi)^2 \hat{u}(\xi, t) + \hat{f}(\xi, t) = -k\xi^2 \hat{u}(\xi, t) + \hat{f}(\xi, t).$$

Taking the Fourier transform of the initial condition $u(x, 0) = g(x)$ gives

$$\hat{u}(\xi, 0) = \hat{g}(\xi).$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ :

$$\hat{u}_t = -k\xi^2 \hat{u} + \hat{f}, \quad \hat{u}(\xi, 0) = \hat{g}(\xi).$$

Applying part (i) with $x = \hat{u}$, $\lambda = -k\xi^2$, $F = \hat{f}$, $G = \hat{g}$ gives

$$\hat{u}(\xi, t) = \hat{g}(\xi) e^{-k\xi^2 t} + \int_0^t e^{-k\xi^2(t-s)} \hat{f}(\xi, s) ds.$$

Therefore

$$u(x, t) = \mathcal{F}^{-1} \left(\hat{g}(\xi) e^{-k\xi^2 t} \right) (x) + \int_0^t \mathcal{F}^{-1} \left(e^{-k\xi^2(t-s)} \hat{f}(\xi, s) \right) (x) ds. \quad (1)$$

Recall that

$$\widehat{e^{-ax^2}}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}.$$

Therefore

$$e^{-k\xi^2 t} = \sqrt{2a} \widehat{e^{-ax^2}}(\xi) \quad \text{for } a = \frac{1}{4kt}.$$

Since the product of Fourier transforms is the Fourier transform of a convolution, we obtain

$$\begin{aligned}\hat{g}(\xi) e^{-k\xi^2 t} &= \sqrt{2a} \hat{g}(\xi) \widehat{e^{-ax^2}}(\xi) \\ &= \sqrt{2a} \frac{1}{\sqrt{2\pi}} g * \widehat{e^{-ax^2}}(\xi) \\ &= \sqrt{\frac{a}{\pi}} g * \widehat{e^{-ax^2}}(\xi) \\ &= \frac{1}{\sqrt{4\pi kt}} g * \widehat{e^{-\frac{x^2}{4kt}}}(\xi)\end{aligned}$$

since $a = \frac{1}{4kt}$. Therefore

$$\mathcal{F}^{-1} \left(\hat{g}(\xi) e^{-k\xi^2 t} \right) (x) = \frac{1}{\sqrt{4\pi kt}} g * e^{-\frac{x^2}{4kt}} = \Phi(\cdot, t) * g = \int_{-\infty}^{\infty} \Phi(x - y, t) g(y) dy \quad (2)$$

where Φ is the fundamental solution of the heat equation in \mathbb{R} . Similarly

$$\mathcal{F}^{-1} \left(e^{-k\xi^2(t-s)} \hat{f}(\xi, s) \right) (x) = \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{x^2}{4k(t-s)}} * f(\cdot, s) = \int_{-\infty}^{\infty} \Phi(x - y, t - s) f(y, s) dy. \quad (3)$$

Combining equations (1), (2) and (3) yields

$$u(x, t) = \int_{-\infty}^{\infty} \Phi(x - y, t) g(y) dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x - y, t - s) f(y, s) dy ds$$

as required.

2. The Fourier transform: The transport equation.

(i) By definition

$$\begin{aligned} \widehat{\tau_a v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau_a v(x) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x - a) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(y) e^{-i\xi(y+a)} dy & (y = x - a) \\ &= e^{-i\xi a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(y) e^{-i\xi y} dy \\ &= e^{-i\xi a} \hat{v}(\xi) \end{aligned}$$

as required.

(ii) Taking the Fourier transform of the transport equation $u_t + cu_x = 0$ gives

$$\hat{u}_t + c i \xi \hat{u} = 0 \quad \Longleftrightarrow \quad \hat{u}_t(\xi, t) + c i \xi \hat{u}(\xi, t) = 0$$

and taking the Fourier transform of the initial condition $u(x, 0) = g(x)$ gives

$$\hat{u}(\xi, 0) = \hat{g}(\xi).$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ :

$$\hat{u}_t = -c i \xi \hat{u}, \quad \hat{u}(\xi, 0) = \hat{g}(\xi).$$

Recall that the ODE $\dot{x} = \lambda x$ has solution $x(t) = x(0)e^{\lambda t}$. Applying this with $x = \hat{u}$, $\lambda = -c i \xi$ yields

$$\begin{aligned} \hat{u}(\xi, t) &= \hat{u}(\xi, 0) e^{-c i \xi t} \\ &= \hat{g}(\xi) e^{-c i \xi t} \\ &= \widehat{\tau_a g}(\xi) \end{aligned}$$

where $a = ct$, by part (i). By taking the inverse Fourier transform we obtain

$$u(x, t) = \tau_a g(x) = g(x - a) = g(x - ct)$$

as desired.

3. *The Fourier transform: Schrödinger's equation.*

- (i) Taking the Fourier transform of $i u_t = -u_{xx}$ with respect to the x variable gives

$$i \widehat{u_t} = -\widehat{u_{xx}} \iff i \hat{u}_t(\xi, t) = -(i\xi)^2 \hat{u}(\xi, t) = \xi^2 \hat{u}(\xi, t).$$

By multiplying by $-i$ we can rewrite this as $\hat{u}_t = -i\xi^2 \hat{u}$. Taking the Fourier transform of the initial condition $u(x, 0) = g(x)$ gives

$$\hat{u}(\xi, 0) = \hat{g}(\xi).$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ :

$$\hat{u}_t = -i\xi^2 \hat{u}, \quad \hat{u}(\xi, 0) = \hat{g}(\xi).$$

Recall that the ODE $\dot{x} = \lambda x$ has solution $x(t) = x(0)e^{\lambda t}$. Applying this with $x = \hat{u}$, $\lambda = -i\xi^2$ gives

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0)e^{-i\xi^2 t} = \hat{g}(\xi)e^{-i\xi^2 t}. \quad (4)$$

To obtain u we need to compute the following inverse Fourier transform:

$$\mathcal{F}^{-1} \left(\hat{g}(\xi)e^{-i\xi^2 t} \right) (x).$$

The trick is to recognise that $\hat{g}(\xi)e^{-i\xi^2 t}$ is the product of Fourier transforms, which follows from the fact that the Fourier transform of a Gaussian is a Gaussian. Recall that

$$\widehat{e^{-ax^2}}(\xi) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}.$$

Therefore

$$e^{-i\xi^2 t} = \sqrt{2a} \widehat{e^{-ax^2}}(\xi) \quad \text{for } a = \frac{1}{4it}. \quad (5)$$

Since the product of Fourier transforms is the Fourier transform of a convolution, we obtain

$$\begin{aligned} \hat{g}(\xi)e^{-i\xi^2 t} &= \sqrt{2a} \hat{g}(\xi) \widehat{e^{-ax^2}}(\xi) && \text{(by equation (5))} \\ &= \sqrt{2a} \frac{1}{\sqrt{2\pi}} g * \widehat{e^{-ax^2}}(\xi) \\ &= \sqrt{\frac{a}{\pi}} g * \widehat{e^{-ax^2}}(\xi). \end{aligned}$$

Combining this with equation (4) and taking the inverse Fourier transform gives

$$\hat{u}(\xi, t) = \sqrt{\frac{a}{\pi}} g * \widehat{e^{-ax^2}}(\xi) \iff u(x, t) = \sqrt{\frac{a}{\pi}} g * e^{-ax^2}.$$

Since $a = \frac{1}{4it}$ and the convolution is commutative we arrive at

$$u(x, t) = \frac{1}{\sqrt{4\pi it}} g * e^{-\frac{x^2}{4it}}(x) = \frac{1}{\sqrt{4\pi it}} e^{\frac{ix^2}{4t}} * g(x) = \frac{1}{\sqrt{4\pi it}} \int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{4t}} g(y) dy$$

as required.

(ii) In part (i) we showed that

$$\hat{u}(\xi, t) = \hat{g}(\xi)e^{-i\xi^2 t}.$$

Since the Fourier transform preserves the L^2 -norm,

$$\begin{aligned}\|u(\cdot, t)\|_{L^2(\mathbb{R})}^2 &= \|\hat{u}(\cdot, t)\|_{L^2(\mathbb{R})}^2 \\ &= \|\hat{g}(\xi)e^{-i\xi^2 t}\|_{L^2(\mathbb{R})}^2 \\ &= \int_{-\infty}^{\infty} |\hat{g}(\xi)e^{-i\xi^2 t}|^2 d\xi \\ &= \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 d\xi \\ &= \|g\|_{L^2(\mathbb{R})}^2\end{aligned}$$

as required.

4. *The Fourier transform: The wave equation.* Use the Fourier transform to derive the solution

$$u(x, t) = \frac{1}{2}[g(x - ct) + g(x + ct)]$$

of the wave equation

$$\begin{aligned}u_{tt} &= c^2 u_{xx} \quad \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) &= g(x) \quad \text{for } x \in \mathbb{R}, \\ u_t(x, 0) &= 0 \quad \text{for } x \in \mathbb{R},\end{aligned}$$

where the constant $c > 0$ is the wave speed. This is known as *D'Alembert's solution*.

Hint: Use Q2(i) and the fact that $\cos(c\xi t) = [\exp(ic\xi t) + \exp(-ic\xi t)]/2$.

Taking the Fourier transform of $u_{tt} = c^2 u_{xx}$ with respect to the x variable gives

$$\widehat{u_{tt}} = \widehat{c^2 u_{xx}} \iff \hat{u}_{tt}(\xi, t) = c^2 (i\xi)^2 \hat{u}(\xi, t) = -c^2 \xi^2 \hat{u}(\xi, t).$$

Taking the Fourier transform of the initial condition $u(x, 0) = g(x)$ gives

$$\hat{u}(\xi, 0) = \hat{g}(\xi).$$

Taking the Fourier transform of the initial condition $u_t(x, 0) = 0$ gives

$$\hat{u}_t(\xi, 0) = 0.$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ :

$$\hat{u}_{tt} = -c^2 \xi^2 \hat{u}, \quad \hat{u}(\xi, 0) = \hat{g}(\xi), \quad \hat{u}_t(\xi, 0) = 0.$$

Recall that the ODE $\ddot{x} = -\lambda^2 x$ has solution of the form $x(t) = A \cos(\lambda t) + B \sin(\lambda t)$. Applying this with $x = \hat{u}$, $\lambda = c\xi$ gives

$$\hat{u}(\xi, t) = A \cos(c\xi t) + B \sin(c\xi t).$$

The initial conditions $\hat{u}(\xi, 0) = \hat{g}(\xi)$, $\hat{u}_t(\xi, 0) = 0$ imply that $A = \hat{g}(\xi)$ and $B = 0$. Therefore

$$\begin{aligned}\hat{u}(\xi, t) &= \hat{g}(\xi) \cos(c\xi t) \\ &= \hat{g}(\xi) \left[\frac{\exp(ic\xi t) + \exp(-ic\xi t)}{2} \right] \\ &= \frac{1}{2} \exp(ic\xi t) \hat{g}(\xi) + \frac{1}{2} \exp(-ic\xi t) \hat{g}(\xi) \\ &= \frac{1}{2} \widehat{\tau_{-a} g}(\xi) + \frac{1}{2} \widehat{\tau_a g}(\xi)\end{aligned}$$

where $a = ct$, by Q2(i). Taking the inverse Fourier transform gives

$$\begin{aligned} u(x, t) &= \frac{1}{2} \tau_{-a} g(x) + \frac{1}{2} \tau_a g(x) \\ &= \frac{1}{2} g(x + a) + \frac{1}{2} g(x - a) \\ &= \frac{1}{2} [g(x + ct) + g(x - ct)] \end{aligned}$$

as required.

5. *The Fourier transform of a derivative.* By definition

$$\begin{aligned} \widehat{u'}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u'(x) e^{-i\xi x} dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \frac{d}{dx} e^{-i\xi x} dx && \text{(integration by parts)} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) (-i\xi) e^{-i\xi x} dx \\ &= i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-i\xi x} dx \\ &= i\xi \widehat{u}(\xi) \end{aligned}$$

as required.

6. *The Fourier transform of a convolution.* By definition

$$\begin{aligned} \widehat{u * v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u * v)(x) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(z) v(x - z) dz \right) e^{-i\xi x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(z) v(x - z) dz \right) e^{-i\xi z} e^{-i\xi(x-z)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(x - z) e^{-i\xi(x-z)} dx \right) u(z) e^{-i\xi z} dz \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(\tilde{x}) e^{-i\xi \tilde{x}} d\tilde{x} \right) u(z) e^{-i\xi z} dz && (\tilde{x} = x - z) \\ &= \int_{-\infty}^{\infty} \widehat{v}(\xi) u(z) e^{-i\xi z} dz \\ &= \sqrt{2\pi} \widehat{v}(\xi) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(z) e^{-i\xi z} dz \\ &= \sqrt{2\pi} \widehat{v}(\xi) \widehat{u}(\xi) \end{aligned}$$

as desired.

7. *Proof of the Sobolev embedding using the Fourier transform.*

(i) Since the Fourier transform preserves the L^2 -norm

$$\begin{aligned}
\|u\|_{H^1(\mathbb{R})}^2 &= \|u\|_{L^2(\mathbb{R})}^2 + \|u'\|_{L^2(\mathbb{R})}^2 \\
&= \|\hat{u}\|_{L^2(\mathbb{R})}^2 + \|\widehat{u'}\|_{L^2(\mathbb{R})}^2 \\
&= \|\hat{u}\|_{L^2(\mathbb{R})}^2 + \|i\xi\hat{u}\|_{L^2(\mathbb{R})}^2 \\
&= \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi + \int_{-\infty}^{\infty} |i\xi\hat{u}(\xi)|^2 d\xi \\
&= \int_{-\infty}^{\infty} (1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi.
\end{aligned}$$

as required.

(ii) Following the hint

$$\begin{aligned}
\|\hat{u}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\hat{u}(\xi)| d\xi \\
&= \int_{-\infty}^{\infty} \frac{1}{(1 + |\xi|^2)^{1/2}} (1 + |\xi|^2)^{1/2} |\hat{u}(\xi)| d\xi \\
&\leq \left(\int_{-\infty}^{\infty} \frac{1}{(1 + |\xi|^2)} d\xi \right)^{1/2} \left(\int_{-\infty}^{\infty} (1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\
&= \left(\int_{-\infty}^{\infty} \frac{1}{(1 + |\xi|^2)} d\xi \right)^{1/2} \|u\|_{H^1(\mathbb{R})} \\
&= C \|u\|_{H^1(\mathbb{R})}
\end{aligned}$$

where

$$C = \left(\int_{-\infty}^{\infty} \frac{1}{(1 + |\xi|^2)} d\xi \right)^{1/2}.$$

If we can show that C is finite, then we've completed the proof. This is a simple calculus exercise; one way is as follows:

$$\begin{aligned}
C^2 &= 2 \int_0^{\infty} \frac{1}{(1 + \xi^2)} d\xi \\
&= \int_0^1 \frac{1}{(1 + \xi^2)} d\xi + \int_1^{\infty} \frac{1}{(1 + \xi^2)} d\xi \\
&\leq \int_0^1 \frac{1}{(1 + 0)} d\xi + \int_1^{\infty} \frac{1}{\xi^2} d\xi \\
&= 1 + 1 \\
&= 2 < \infty
\end{aligned}$$

as required.

(iii) By the Fourier Inversion Theorem

$$\begin{aligned}
|u(x)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{i\xi x} d\xi \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{u}(\xi)| |e^{i\xi x}| d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{u}(\xi)| d\xi \\
&= \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L^1(\mathbb{R})} \\
&\leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^1(\mathbb{R})}
\end{aligned}$$

by part (ii). Since this holds for all $x \in \mathbb{R}$,

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^1(\mathbb{R})}$$

as required.

8. *Fundamental Solution of the Heat Equation.* We compute

$$\begin{aligned}
\Phi_t(\mathbf{x}, t) &= \frac{1}{(4\pi k)^{\frac{n}{2}}} e^{-\frac{|\mathbf{x}|^2}{4kt}} \left(-\frac{n}{2} t^{-\frac{n}{2}-1} + \frac{|\mathbf{x}|^2}{4kt^2} \right) \\
\frac{\partial \Phi}{\partial x_i}(\mathbf{x}, t) &= \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|\mathbf{x}|^2}{4kt}} \left(-\frac{2x_i}{4kt} \right), \\
\frac{\partial^2 \Phi}{\partial x_i^2}(\mathbf{x}, t) &= \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|\mathbf{x}|^2}{4kt}} \left(-\frac{2}{4kt} + \left(\frac{2x_i}{4kt} \right)^2 \right) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|\mathbf{x}|^2}{4kt}} \left(-\frac{1}{2kt} + \frac{x_i^2}{4k^2 t^2} \right), \\
\Delta \Phi(\mathbf{x}, t) &= \sum_{i=1}^n \frac{\partial^2 \Phi}{\partial x_i^2}(\mathbf{x}, t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|\mathbf{x}|^2}{4kt}} \left(-\frac{n}{2kt} + \frac{|\mathbf{x}|^2}{4k^2 t^2} \right).
\end{aligned}$$

Therefore Φ satisfies $\Phi_t(\mathbf{x}, t) = k\Delta\Phi(\mathbf{x}, t)$ for all $\mathbf{x} \in \mathbb{R}^n$, $t > 0$, as required.

9. *Finite speed of propagation for a degenerate diffusion equation.*

(i) Observe that

$$\frac{1}{2} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} - \frac{1}{6k} \frac{x^2}{t} = 0 \iff |x| = 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$$

and so

$$\Phi(x, t) = \begin{cases} \frac{1}{2} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} - \frac{1}{6k} \frac{x^2}{t} & \text{if } |x| < 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}, \\ 0 & \text{if } |x| \geq 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}. \end{cases}$$

Clearly Φ satisfies the degenerate diffusion equation for $|x| > 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$. For $|x| < 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$,

$$\begin{aligned}
\Phi_t(x, t) &= \frac{1}{6} \left(\frac{3}{k\pi t} \right)^{-\frac{2}{3}} \left(-\frac{3}{k\pi t^2} \right) + \frac{x^2}{6kt^2} = -\frac{1}{6t} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} + \frac{x^2}{6kt^2}, \\
\Phi_x(x, t) &= -\frac{x}{3kt},
\end{aligned}$$

$$(a(\Phi)\Phi_x)(x, t) = -\frac{x}{6t} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} + \frac{x^3}{18kt^2},$$

$$(a(\Phi)\Phi_x)_x(x, t) = -\frac{1}{6t} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} + \frac{x^2}{6kt^2}.$$

Therefore Φ satisfies the degenerate diffusion equation for all $t > 0$ and all $x \in \mathbb{R}$ with $|x| \neq 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$.

- (ii) For fixed $t > 0$, $\Phi(x, t)$ is the maximum of a concave quadratic function and the zero function. The support of the map $x \mapsto \Phi(x, t)$ is the compact interval

$$[-3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}, 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}].$$

10. *The mathematical equation that caused the banks to crash.* Let $t(x, \tau)$ satisfy

$$\tau = T - t(x, \tau).$$

Differentiating this expression with respect to τ and x gives

$$t_\tau = -1, \quad t_x = 0.$$

Let $S(x, \tau)$ satisfy

$$x = \ln \left(\frac{S(x, \tau)}{K} \right) + \left(r - \frac{1}{2} \sigma^2 \right) \tau. \quad (6)$$

Differentiating this expressions with respect to τ gives

$$0 = \frac{K}{S} \frac{S_\tau}{K} + r - \frac{1}{2} \sigma^2 \iff S_\tau = \left(\frac{1}{2} \sigma^2 - r \right) S.$$

Differentiating equation (6) with respect to x gives

$$1 = \frac{K}{S} \frac{S_x}{K} \iff S_x = S.$$

Therefore

$$\begin{aligned} u_\tau &= Ce^{r\tau} [rV + V_S S_\tau + V_t t_\tau] = Ce^{r\tau} \left[rV + \left(\frac{1}{2} \sigma^2 - r \right) S V_s - V_t \right], \\ u_x &= Ce^{r\tau} [V_S S_x + V_t t_x] = Ce^{r\tau} S V_s, \\ u_{xx} &= Ce^{r\tau} [S_x V_s + S V_{SS} S_x] = Ce^{r\tau} [S V_s + S^2 V_{SS}]. \end{aligned}$$

Therefore

$$\begin{aligned} u_\tau - \frac{1}{2} \sigma^2 u_{xx} &= Ce^{r\tau} \left[rV + \left(\frac{1}{2} \sigma^2 - r \right) S V_s - V_t - \frac{1}{2} \sigma^2 (S V_s + S^2 V_{SS}) \right] \\ &= -Ce^{r\tau} \left[V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_s - r V \right] \\ &= 0 \end{aligned}$$

since V satisfies the Black-Scholes PDE. This completes the proof.

11. *The energy method: Uniqueness for the heat equation in a time dependent domain.* Let u and v be solutions and let $w = u - v$. Then w satisfies

$$\begin{aligned} w_t - kw_{xx} &= 0 & \text{for } (x, t) \in U, \\ w(a(t), t) &= 0 & \text{for } t \in [0, T], \\ w(b(t), t) &= 0 & \text{for } t \in [0, T], \\ w(x, 0) &= 0 & \text{for } x \in (a(0), b(0)). \end{aligned}$$

Multiply the equation $w_t = kw_{xx}$ by w and integrate over $(a(t), b(t))$ to obtain

$$\int_{a(t)}^{b(t)} ww_t dx = k \int_{a(t)}^{b(t)} ww_{xx} dx. \quad (7)$$

Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \int_{a(t)}^{b(t)} f_t(x, t) dx + \dot{b}(t)f(b(t), t) - \dot{a}(t)f(a(t), t).$$

We can use this to rewrite the left-hand side of equation (7) as follows:

$$\begin{aligned} \int_{a(t)}^{b(t)} ww_t dx &= \frac{d}{dt} \frac{1}{2} \int_{a(t)}^{b(t)} w^2(x, t) dx - \frac{1}{2} \dot{b}(t)w^2(b(t), t) + \frac{1}{2} \dot{a}(t)w^2(a(t), t) \\ &= \frac{d}{dt} \frac{1}{2} \int_{a(t)}^{b(t)} w^2(x, t) dx \end{aligned} \quad (8)$$

since $w(a(t), t) = w(b(t), t) = 0$. We rewrite the right-hand side of equation (7) using integration by parts:

$$k \int_{a(t)}^{b(t)} ww_{xx} dx = kw w_x \Big|_{a(t)}^{b(t)} - k \int_{a(t)}^{b(t)} w_x^2 dx = -k \int_{a(t)}^{b(t)} w_x^2 dx, \quad (9)$$

again using the fact that $w(a(t), t) = w(b(t), t) = 0$. Substituting (8) and (9) into (7) gives

$$\frac{d}{dt} \frac{1}{2} \int_{a(t)}^{b(t)} w^2(x, t) dx = -k \int_{a(t)}^{b(t)} w_x^2 dx \leq 0.$$

Let

$$E(t) = \frac{1}{2} \int_{a(t)}^{b(t)} w^2(x, t) dx.$$

We have shown that $\dot{E}(t) \leq 0$. Hence

$$0 \leq E(t) \leq E(0) = 0$$

since $w = 0$ for $t = 0$. Consequently $E(t) = 0$ for all t . Therefore $w = 0$ in U and so $u = v$, as required.

12. *The energy method: Uniqueness for a 4th-order heat equation.* Let u and v be solutions and let $w = u - v$. Then w satisfies

$$w_t + kw_{xxxx} = 0 \quad \text{for } (x, t) \in (a, b) \times (0, T], \quad (10)$$

$$w(a, t) = w(b, t) = 0 \quad \text{for } t \in [0, T], \quad (11)$$

$$w_x(a, t) = w_x(b, t) = 0 \quad \text{for } t \in [0, T], \quad (12)$$

$$w(x, 0) = 0 \quad \text{for } x \in (a, b). \quad (13)$$

Multiplying the equation $w_t = -kw_{xxxx}$ by w and integrating over (a, b) gives

$$\begin{aligned}
\int_a^b \underbrace{ww_t}_{\frac{1}{2} \frac{\partial}{\partial t} w^2} dx &= -k \int_a^b ww_{xxxx} dx \\
&= -kw_{xxx} \Big|_a^b + k \int_a^b w_x w_{xxx} dx \\
&= k \int_a^b w_x w_{xxx} dx && \text{(by (11))} \\
&= kw_x w_{xx} \Big|_a^b - k \int_a^b w_{xx} w_{xx} dx \\
&= -k \int_a^b w_{xx}^2 dx && \text{(by (12)).}
\end{aligned}$$

Therefore

$$\frac{d}{dt} \frac{1}{2} \int_a^b w^2 dx = -k \int_a^b w_{xx}^2 dx \leq 0$$

and so

$$0 \leq \int_a^b w^2(x, t) dx \leq \int_a^b w^2(x, 0) dx = 0$$

by (13). We conclude that $w = 0$ and hence $u = v$, as required.

We consider $u_t + ku_{xxxx} = 0$ to be the 4th-order version of the heat equation $u_t - ku_{xx} = 0$ since it has the same energy-decay property:

$$\frac{d}{dt} \|u\|_{L^2([a, b])}^2 \leq 0$$

(provided that u and u_x vanish at $x = a$ and $x = b$). The equation $u_t - ku_{xxxx} = 0$ looks more similar to the heat equation $u_t - ku_{xx} = 0$ (because of the minus sign), but its L^2 -energy grows with time:

$$\frac{d}{dt} \|u\|_{L^2([a, b])}^2 \geq 0.$$

13. *Asymptotic behaviour of the heat equation with time independent data.* Let $w(\mathbf{x}, t) = u(\mathbf{x}, t) - v(\mathbf{x})$. We need to prove that $\lim_{t \rightarrow \infty} \|w\|_{L^2(\Omega)} = 0$. By subtracting the PDEs for u and v we find that w satisfies

$$\begin{aligned}
w_t(\mathbf{x}, t) - k\Delta w(\mathbf{x}, t) &= 0 && \text{for } (\mathbf{x}, t) \in \Omega \times (0, \infty), \\
w(\mathbf{x}, t) &= 0 && \text{for } (\mathbf{x}, t) \in \partial\Omega \times [0, \infty), \\
w(\mathbf{x}, 0) &= u_0(\mathbf{x}) - v(\mathbf{x}) && \text{for } \mathbf{x} \in \Omega.
\end{aligned}$$

Multiplying the equation $w_t = k\Delta w$ by w and integrating by parts over Ω gives

$$\begin{aligned}
\int_{\Omega} ww_t d\mathbf{x} &= k \int_{\Omega} w\Delta w d\mathbf{x} \iff \frac{d}{dt} \frac{1}{2} \int_{\Omega} w^2 d\mathbf{x} = k \int_{\partial\Omega} w \nabla w \cdot \mathbf{n} dS - k \int_{\Omega} \nabla w \cdot \nabla w d\mathbf{x} \\
&\iff \frac{d}{dt} \frac{1}{2} \int_{\Omega} w^2 d\mathbf{x} = -k \int_{\Omega} |\nabla w|^2 d\mathbf{x} && (14)
\end{aligned}$$

since $w = 0$ on $\partial\Omega$. By the Poincaré inequality, there exists a constant $C_p > 0$ such that

$$\int_{\Omega} |w|^2 d\mathbf{x} \leq C_p \int_{\Omega} |\nabla w|^2 d\mathbf{x}.$$

Multiplying this by $-k/C_p$ gives

$$-\frac{k}{C_p} \int_{\Omega} |w|^2 d\mathbf{x} \geq -k \int_{\Omega} |\nabla w|^2 d\mathbf{x}. \quad (15)$$

Combining equations (14), (15) yields

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} w^2 d\mathbf{x} \leq -\frac{k}{C_p} \int_{\Omega} |w|^2 d\mathbf{x}.$$

Define

$$E(t) = \int_{\Omega} w^2(\mathbf{x}, t) d\mathbf{x} = \|w\|_{L^2(\Omega)}^2$$

and $\lambda = \frac{2k}{C_p}$. We have shown that

$$\dot{E} \leq -\lambda E.$$

By the Grönwall inequality,

$$E(t) \leq e^{-\lambda t} E(0).$$

Since $\lambda > 0$, we conclude that $E(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore $w \rightarrow 0$ in $L^2(\Omega)$ as $t \rightarrow \infty$, as required.

14. *Asymptotic behaviour of the heat equation with time independent data in the L^∞ -norm.*

(i) By linearity, w satisfies

$$\begin{aligned} w_t(x, t) - kw_{xx}(x, t) &= 0 \quad \text{for } (x, t) \in (a, b) \times (0, \infty), \\ w(x, 0) &= u_0(x) - v(x) \quad \text{for } x \in (a, b), \\ w(a, t) &= w(b, t) = 0 \quad \text{for } t \in [0, \infty). \end{aligned}$$

Multiplying the PDE by w and integrating over $[a, b]$ gives

$$\int_a^b ww_t dx = k \int_a^b ww_{xx} dx \iff \frac{d}{dt} \frac{1}{2} \int_a^b w^2 dx = \underbrace{kww_x \Big|_a^b}_{=0} - k \int_a^b w_x^2 dx$$

where we have used the Chain Rule and integration by parts. Note that the boundary terms vanish when we perform integration by parts since $w(a, t) = w(b, t) = 0$. Therefore

$$\frac{d}{dt} \int_a^b w^2(x, t) dx = -2k \int_a^b w_x^2(x, t) dx$$

as required.

(ii) We can write the result of part (i) in terms of L^2 -norms as

$$\frac{d}{dt} \|w\|_{L^2([a, b])}^2 = -2k \|w_x\|_{L^2([a, b])}^2. \quad (16)$$

By the Poincaré inequality, there exists a constant $C > 0$ such that

$$\|w\|_{L^2([a, b])}^2 \leq C \|w_x\|_{L^2([a, b])}^2. \quad (17)$$

Combining equations (16), (17) gives

$$\frac{d}{dt} \|w\|_{L^2([a, b])}^2 = -2k \|w_x\|_{L^2([a, b])}^2 \leq -\frac{2k}{C} \|w\|_{L^2([a, b])}^2. \quad (18)$$

Define

$$E(t) = \|w\|_{L^2([a,b])}^2.$$

We can rewrite equation (18) as

$$\dot{E} \leq -\lambda E$$

with $\lambda = 2k/C > 0$. By the Grönwall inequality,

$$E(t) \leq E(0)e^{-\lambda t} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Therefore, by definition of E , $w \rightarrow 0$ in $L^2([a,b])$ as $t \rightarrow \infty$, as required.

(iii) By differentiating the PDE for w with respect to t we obtain

$$\begin{aligned} w_{tt}(x,t) - kw_{txx}(x,t) &= 0 \quad \text{for } (x,t) \in (a,b) \times (0,\infty), \\ w_t(a,t) = w_t(b,t) &= 0 \quad \text{for } t \in [0,\infty). \end{aligned}$$

In particular, w_t satisfies the heat equation with Dirichlet boundary conditions, just like w . Therefore the argument we applied in parts (i) and (ii) to w can also be applied to w_t , which yields $w_t \rightarrow 0$ in $L^2([a,b])$ as $t \rightarrow \infty$.

(iv) We have

$$\begin{aligned} \|w_x\|_{L^2([a,b])}^2 &= \int_a^b w_x^2(x,t) dx \\ &= -\frac{1}{2k} \frac{d}{dt} \int_a^b w^2(x,t) dx && \text{(by part (i))} \\ &= -\frac{1}{k} \int_a^b w(x,t) w_t(x,t) dx \\ &\leq \frac{1}{k} \left(\int_a^b w^2(x,t) dx \right)^{1/2} \left(\int_a^b w_t^2(x,t) dx \right)^{1/2} && \text{(Cauchy-Schwarz)} \\ &= \frac{1}{k} \|w\|_{L^2([a,b])} \|w_t\|_{L^2([a,b])} \rightarrow 0 \quad \text{as } t \rightarrow \infty \end{aligned}$$

by parts (ii) and (iii). Therefore $w_x \rightarrow 0$ in $L^2([a,b])$ as $t \rightarrow \infty$, as required. Note that we don't really need $w_t \rightarrow 0$ in $L^2([a,b])$ as $t \rightarrow \infty$, we just need $\|w_t\|_{L^2([a,b])}$ to be uniformly bounded in t .

(v) This final result follows from the Sobolev inequality: There exists a constant $C > 0$ such that

$$\|w\|_{L^\infty([a,b])} \leq C \|w\|_{H^1([a,b])} = C \left(\|w\|_{L^2([a,b])} + \|w_x\|_{L^2([a,b])} \right)^{1/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

by parts (ii) and (iv).

15. Applications of the maximum principle: Uniqueness and bounds on solutions.

(i) Let $\Gamma_T = [a,b] \times \{0\} \cup \{a,b\} \times [0,T]$ be the parabolic boundary of Ω_T . Let $u, v \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ satisfy

$$\begin{aligned} u_t - u_{xx} &= 1 && \text{in } \Omega_T, \\ u &= 0 && \text{in } \Gamma_T. \end{aligned}$$

Then $w = u - v \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 && \text{in } \Omega_T, \\ w &= 0 && \text{in } \Gamma_T. \end{aligned}$$

By the weak maximum principle

$$\max_{\overline{\Omega}_T} w = \max_{\Gamma_T} w = 0, \quad \min_{\overline{\Omega}_T} w = \min_{\Gamma_T} w = 0.$$

Therefore $w = 0$ and $u = v$, as required.

(ii) Since $u_t - u_{xx} = 1 > 0$, the weak maximum principle gives

$$\min_{\overline{\Omega}_T} u = \min_{\Gamma_T} u = 0.$$

This is the desired lower bound on u . We still need to prove the upper bound. Let $v(x, t) = t$. Then $v_t - v_{xx} = 1$ and $w = u - v$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 & \text{in } \Omega_T, \\ w &= -t & \text{in } \Gamma_T. \end{aligned}$$

By the weak maximum principle

$$\max_{\overline{\Omega}_T} w = \max_{\Gamma_T} w = \max_{\Gamma_T} (-t) = 0.$$

Therefore $w \leq 0$ in Ω_T and hence $u \leq v = t$ in Ω_T , which is the desired upper bound.

16. *Application of the maximum principle: Comparison Principle.* Define $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t}(\mathbf{x}, t) - k\Delta v(\mathbf{x}, t) &= f_1(\mathbf{x}) - f_2(\mathbf{x}) & \text{for } (\mathbf{x}, t) \in \Omega \times (0, T], \\ v(\mathbf{x}, t) &= g_1(\mathbf{x}) - g_2(\mathbf{x}) & \text{for } (\mathbf{x}, t) \in \partial\Omega \times [0, T], \\ v(\mathbf{x}, 0) &= u_1^0(\mathbf{x}) - u_2^0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{aligned}$$

Since $f_1 \leq f_2$, then

$$v_t - k\Delta v = f_1 - f_2 \leq 0 \quad \text{in } \Omega_T.$$

Therefore the weak maximum principle implies that

$$\max_{\overline{\Omega}_T} v = \max_{\Gamma_T} v.$$

For $(\mathbf{x}, t) \in \Gamma_T$,

$$v(\mathbf{x}, t) = \begin{cases} g_1(\mathbf{x}) - g_2(\mathbf{x}) & \text{if } (\mathbf{x}, t) \in \partial\Omega \times [0, T], \\ u_1^0(\mathbf{x}) - u_2^0(\mathbf{x}) & \text{if } t = 0, \mathbf{x} \in \Omega. \end{cases}$$

But

$$g_1 - g_2 \leq 0, \quad u_1^0 - u_2^0 \leq 0.$$

Therefore $v \leq 0$ on Γ_T and hence

$$\max_{\overline{\Omega}_T} v = \max_{\Gamma_T} v \leq 0.$$

Hence $v \leq 0$ in Ω_T and so $u_1 \leq u_2$ in Ω_T , as required.

17. *Eigenfunctions of the Laplacian and an application to the heat equation.* Formally (not worrying about interchanging limits and infinite sums),

$$\begin{aligned} 0 &= v_t - k\Delta v \\ &= \sum_{n=1}^{\infty} \dot{c}_n(t) u_n(\mathbf{x}) - k \sum_{n=1}^{\infty} c_n(t) \Delta u_n(\mathbf{x}) \\ &= \sum_{n=1}^{\infty} \dot{c}_n(t) u_n(\mathbf{x}) + k \sum_{n=1}^{\infty} c_n(t) \lambda_n u_n(\mathbf{x}) \\ &= \sum_{n=1}^{\infty} (\dot{c}_n(t) + k\lambda_n c_n(t)) u_n(\mathbf{x}). \end{aligned}$$

Since $\{u_n\}_{n \in \mathbb{N}}$ forms an orthogonal basis, it follows that

$$\dot{c}_n(t) + k\lambda_n c_n(t) = 0$$

for all n . We also have

$$v(\mathbf{x}, 0) = g(\mathbf{x}) \iff \sum_{n=1}^{\infty} c_n(0) u_n(\mathbf{x}) = \sum_{n=1}^{\infty} g_n u_n(\mathbf{x}) \iff \sum_{n=1}^{\infty} (c_n(0) - g_n) u_n(\mathbf{x}) = 0.$$

Again, since $\{u_n\}_{n \in \mathbb{N}}$ forms an orthogonal basis, it follows that

$$c_n(0) = g_n$$

for all n . We have reduced the PDE for v to a one-parameter family of uncoupled ODEs, indexed by n :

$$\dot{c}_n(t) = -k\lambda_n c_n(t), \quad c_n(0) = g_n.$$

These ODEs have solutions

$$c_n(t) = g_n e^{-k\lambda_n t}.$$

Therefore

$$v(\mathbf{x}, t) = \sum_{n=1}^{\infty} g_n e^{-k\lambda_n t} u_n(\mathbf{x})$$

as required.