Partial Differential Equations III & V, Additional Exercise - Wave Equation

1. Consider the wave equation

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & (x,t) \in (0,2\pi) \times (0,\infty)\,, \\ u(0,t) = u(2\pi,t), & t \in (0,\infty), \\ u(x,0) = f(x), & t \in (0,2\pi), \\ u_t\,(x,0) = 0, & x \in (0,2\pi)\,, \end{cases} \tag{1}$$

where f is a smooth function on $[0,2\pi]$ with $f(0) = f(2\pi) = 0$. We will find a solution to the (1) by utilising the Fourier series (like the heat equation). We assume that we can write

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(t) e^{inx}$$

with

$$a_n(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx} dx$$

and that the convergence is such that all interchanging of differentiation and integration is allowed (this will happen, for instance, if we seek a solution that is C^4 on the periodic domain).

(i) Show that $a_n(t)$ satisfies the equation

$$a_n''(t) + n^2 c^2 a_n(t) = 0.$$

(ii) using he boundary conditions show that

$$a_n(t) = A_n \cos(nct)$$

and find an explicit expression for $\{A_n\}_{n\in\mathbb{Z}}$ which depends on f.

(iii) Write a solution to (1)