Partial Differential Equations III & V, Additional Exercise - Wave Equation Solution

1. (i) Differentiating the sum and plugging it into our wave equation yields

$$\sum_{n\in\mathbb{Z}} a_n''(t)e^{inx} = c^2 \sum_{n\in\mathbb{Z}} a_n(t)\,(in)^2\, e^{inx}.$$

The uniqueness of the Fourier coefficients implies that $a_n(t)$ must solve the equation

$$a_n''(t) = -n^2c^2a_n(t),$$

which is the desired result.

(ii) The solution to the above sequence of ODEs is given by

$$a_0(t) = A_0 + B_0t,$$

and

$$a_n(t) = A_n \cos(nct) + B_n \sin(nct)$$

when $n \neq 0$. At this point we will need to use our boundary condition for $u(x,0)$ and $u_t(x,0)$. Since

$$u_t(x,t) = \sum_{n\in\mathbb{Z}} a_n'(t)e^{inx}$$

we conclude that $u_t(x,0) = 0$ implies, together with the uniqueness of the Fourier coefficients, that

$$a_n'(0) = 0, \qquad \forall n \in \mathbb{Z}.$$

As

$$a_0'(t) = B_0$$

and

$$a_n'(t) = -ncA_n \sin(nct) + ncB_n \cos(nct),$$

when $n \neq 0$ we find that $B_n = 0$ for all $n \in \mathbb{Z}$. Thus, we have that

$$a_n(t) = A_n \cos(nct), \qquad n \in \mathbb{Z}.$$

Using the fact that $u(x,0) = f(x)$ we find that

$$\sum_{n\in\mathbb{Z}} a_n(0)e^{inx} = u(x,0) = f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{inx}$$

and, like before, the uniqueness of the Fourier coefficients imply that

$$A_n = a_n(0) = \widehat{f}_n, \qquad \forall n \in \mathbb{Z}.$$

(iii) Combining the previous results we conclude that our solution is given by

$$u(x,t) = \sum_{n\in\mathbb{Z}} \widehat{f}_n \cos(nct)\, e^{inx}$$