Partial Differential Equations III – Exam 2022/23

Section A

1. Let us consider the following Cauchy problem associated to a first order PDE

$$\begin{cases} x_2 \partial_{x_1} u(x_1, x_2) = 1, & (x_1, x_2) \in \mathbb{R}^2, \\ u(0, x_2) = 0, & x_2 \in \mathbb{R}. \end{cases} \tag{1}$$

   (a) Identify the leading vector field, the Cauchy data and the Cauchy curve.

   (b) Are the points on the Cauchy curve characteristic or non-characteristic? Justify your answer.

   (c) Using the method of characteristics, solve the problem in (1). Give the domain of definition of the solution.

2. Consider the Cauchy problem for Burgers' equation

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2}\partial_x(u^2(x,t)) = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

   where $u_0 : \mathbb{R} \to \mathbb{R}$ is given.

   (a) Let $u_0(x) = \frac{1}{7}x^7$. Show that (2) has a global in time classical solution.

   (b) Let $u_0(x) = \sin(x)$. Write down the definition of the critical time t_c (until when we can guarantee the existence of a classical solution to (2)) associated to this initial datum. Show that $t_c \leq 1$.

3. In this problem we consider harmonic function on the unit ball in $\mathbb{R}^3$, $B_1(0)$.

   (a) Using the fact that the Laplacian in spherical coordinates is given by

$$\Delta\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

   show that if $u \in C^2\left(\overline{B_1(0)}\right)$ is radial (i.e. only depends on r in spherical coordinates) and harmonic in $B_1(0)$ then it must be constant.

   (b) Show that there exists no radial solution in $C^2\left(\overline{B_1(0)}\right)$ to the equation

$$\begin{cases} -\Delta u(\vec{x}) = 0, & \vec{x} \in B_1(0), \\ u(\vec{x}) = f(\vec{x}), & \vec{x} \in \partial B_1(0), \end{cases}$$

   for $f(\vec{x}) = x_1^2$.

4. Consider the heat-like equation

$$\begin{cases} u_t - u_{xx} + cu = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where $c \in \mathbb{R}$ is a fixed constant and $g \in C_c(\mathbb{R})$.

(a) Define $v(x,t) = e^{ct}u(x,t)$. Show that $v(x,t)$ solves the heat equation

$$\begin{cases} v_t - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ v(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(b) Show that there exists a solution to (3) that satisfies

$$\sup_{x \in \mathbb{R}} |u(x,t)| \leq e^{-ct} \|g\|_{L^\infty(\mathbb{R})}.$$

You may use the following inequality without proof: For any $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ we have that

$$\left| \int_{\mathbb{R}} f(x-y)\, g(y) \right| \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})}, \qquad \forall x \in \mathbb{R}.$$

Section B

5. We consider the following conservation law

$$\begin{cases} \partial_t u(x,t) - u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$

[Notice that this is *not* Burgers' equation.]

(a) Suppose that u_0 is bounded, differentiable with bounded derivative. Give a formula of the critical time t_c, for which we know that (4) has a classical solution on $\mathbb{R} \times (0, t_c)$.

(b) Let $u_0(x) = -\arctan(x)$. Show that in this case (4) has a global in time classical solution.

(c) Let u_0 now be given by

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

By drawing the characteristics, show that there is instantaneous crossing of characteristics. Find a shock that satisfies the Rankine–Hugoniot condition. Give the expression of the weak solution in this case.

6. We aim to solve the following problem by the method of characteristics

$$\begin{cases} \partial^2_{xx} u - 3\partial^2_{xy} u + 2\partial^2_{yy} u = 0, & (x,y) \in \mathbb{R}^2, \\ u(1,y) = g(y), & y \in \mathbb{R}, \\ \partial_x u(1,y) = h(y), & y \in \mathbb{R}, \end{cases} \tag{5}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are given smooth functions.

(a) Identify the Cauchy data and the Cauchy curve in the above problem.

(b) Rewrite the PDE in (5) as a system of two linear first order PDEs. [*Hint*: think about the algebraic relation $(a-b)(a-2b) = a^2 - 3ab + 2b^2$, $(a, b \in \mathbb{R})$.]

(c) By solving the two first order PDEs arising from 6b using the method of characteristics, find the solution to (5).

7. Let Ω be an open bounded set with smooth boundary in $\mathbb{R}^n$ and let u_1 and u_2 be $C^2(\Omega) \cap C(\overline{\Omega})$ solutions to Poisson equation

$$\begin{cases} -\Delta u_i(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u_i(\vec{x}) = g_i(\vec{x}), & \vec{x} \in \partial\Omega, \end{cases}$$

$i = 1, 2$, where $f \in C^1(\overline{\Omega})$ and $g_1, g_2 \in C(\partial\Omega)$.

(a) Show that for any $\vec{x} \in \overline{\Omega}$

$$u_2(\vec{x}) - u_1(\vec{x}) \leq \max_{\vec{x} \in \partial\Omega} (g_2(\vec{x}) - g_1(\vec{x})).$$

(b) Show that

$$\max_{\vec{x} \in \overline{\Omega}} |u_2(\vec{x}) - u_1(\vec{x})| \leq \max_{\vec{x} \in \partial\Omega} |g_2(\vec{x}) - g_1(\vec{x})|.$$

(c) For $n \in \mathbb{N}$ let $u_n \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the system

$$\begin{cases} -\Delta u_n(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u_n(\vec{x}) = g_n(\vec{x}), & \vec{x} \in \partial\Omega, \end{cases}$$

and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the system

$$\begin{cases} -\Delta u(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u(\vec{x}) = g(\vec{x}), & \vec{x} \in \partial\Omega. \end{cases}$$

Show that if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $\partial\Omega$ then $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to u on $\overline{\Omega}$.

Recall that we say that a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ in $C(K)$ converges uniformly to $f \in C(K)$ if

$$\sup_{x \in K} |f_n(x) - f(x)| \underset{n \to \infty}{\longrightarrow} 0.$$

8. Let $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a classical solution to the equation

$$\begin{cases} u_t + k u_{xxxx} = 0, & (x, t) \in \mathbb{R} \times (0, +\infty), \\ u(x, 0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $k > 0$ is a fixed constant and f is a smooth function on $\mathbb{R}$ that belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(a) Show that $\widehat{u}$, the Fourier transform of u in the $x-$variable, satisfies

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) e^{-k\xi^4 t}.$$

(b) Using the fact that the Fourier transform preserves the L^2 norm (Plancherel's identity) as well as the fact that it is in L^∞ to show that

$$\|u(\cdot, t)\|^2_{L^2(\mathbb{R})} \leq \frac{\left(\int_{\mathbb{R}} e^{-x^4} dx\right)^{\frac{1}{2}}}{\sqrt[8]{4kt}} \|f\|_{L^1(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

9. For $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = \sqrt{x^2 + 1/n}$.

(a) Show that as $n \to +\infty$, the sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$.

(b) Compute the pointwise limit of the sequence $(f_n')_{n \in \mathbb{N}}$ as $n \to +\infty$. Can this limit be the uniform limit of the sequence? Justify your answer.

(c) Show that as $n \to +\infty$, the sequence $(f_n'')_{n \in \mathbb{N}}$ converges in the sense of distributions to $2\delta_0$, where δ_0 is the Dirac mass concentrated at 0.