Partial Differential Equations III – Exam 2022/23 Solution

Section A

1. Let us consider the following Cauchy problem associated to a first order PDE

$$\begin{cases} x_2\partial_{x_1}u(x_1,x_2) = 1, & (x_1,x_2)\in\mathbb{R}^2, \\ u(0,x_2) = 0, & x_2\in\mathbb{R}. \end{cases} \tag{1}$$

(a) Identify the leading vector field, the Cauchy data and the Cauchy curve.

(b) Are the points on the Cauchy curve characteristic or non-characteristic? Justify your answer.

(c) Using the method of characteristics, solve the problem in (1). Give the domain of definition of the solution.

Solution:

1.(a) The leading vector field $\vec{a}:\mathbb{R}^2\to\mathbb{R}^2$ is given by $\vec{a}(x_1,x_2) = (x_2,0)$, the Cauchy datum is $u(0,x_2) = 0$ and the Cauchy curve is given by $\{\gamma(s) := (0,s) : s\in\mathbb{R}\}\subset\mathbb{R}^2$, which is in fact the x_2-axis.

1.(b) We have $\gamma'(s) = (0,1)$, and we have $\vec{a}(x_1,x_2)\cdot\gamma'(s) = 0$, so the vector field is always orthogonal to the Cauchy curve. The only 'problematic' point is the origin, as $\vec{a}(0,0)$ vanishes, and therefore all points on the Cauchy curve, except $(0,0)$, are non-characteristic. The point $(0,0)$ is characteristic and we expect the solutions to have problems at this point.

1.(c) We have the system of ODEs (where $\tau\in\mathbb{R}$ stands for the artificial time parameter and $s\in\mathbb{R}$ is used for the parametrisation of the Cauchy curve)

$$\begin{cases} \partial_\tau x_1(\tau,s) = x_2(\tau,s), \\ \partial_\tau x_2(\tau,s) = 0, \\ \partial_\tau z(\tau,s) = 1 \end{cases}$$

equipped with the boundary conditions

$$x_1(0,s) = 0, \quad x_2(0,s) = s, \quad z(0,s) = 0.$$

From here, we directly find

$$x_1(\tau,s) = \tau s, \quad x_2(\tau,s) = s, \quad z(\tau,s) = \tau.$$

For a generic point $(X_1,X_2)\in\mathbb{R}^d$, we find $s = X_2$ and $\tau = X_1/X_2$, provided $X_2\neq 0$. Therefore the solution becomes $u(x_1,x_2) = x_1/x_2$ and the domain of this function is $\mathbb{R}^2\setminus\{(x_1,0) : x_1\in\mathbb{R}\}$. We notice that since $\vec{a}(0,0) = (0,0)$, no information can be transported along the x_1-axis.

2. Consider the Cauchy problem for Burgers' equation

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2}\partial_x(u^2(x,t)) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is given.

(a) Let $u_0(x) = \frac{1}{7}x^7$. Show that (2) has a global in time classical solution.

(b) Let $u_0(x) = \sin(x)$. Write down the definition of the critical time t_c (until when we can guarantee the existence of a classical solution to (2)) associated to this initial datum. Show that $t_c \le 1$.

Solution: Some general fact used in both questions below. In the case of Burgers' equation, we have that the wave speed is given by $c(u) = u$ and the characteristics are given in the (x,t)-(half)plane $(t \ge 0)$ by

$$x = s + u_0(s)t,$$

where $s \in \mathbb{R}$ parametrises the characteristics.

2.(a) If $u_0 \in C^1(\mathbb{R})$, the global existence of a classical solution to Burgers' boils down to the global invertibility of the flow map $s \mapsto s + u_0(s)t$, for every $t > 0$. A sufficient condition for this is $1+u_0'(s)t > 0$ for all $t > 0$ and for all $s \in \mathbb{R}$. As u_0 is increasing, we have $1+u_0'(s)t = 1+ts^6 > 0$ for all $t > 0$ and all $s \in \mathbb{R}$. This implies the global in time existence of a classical solution.

2.(b) The critical time is defined as the greatest $t > 0$ such that we can guarantee the global invertibility of the flow map. This is formally defined as

$$t_c := \inf\left\{-\frac{1}{u_0'(s)} : s \in I\right\},$$

where $I = \{s \in \mathbb{R} : u_0'(s) < 0\}$.
In this case $u_0'(s) = \cos(s)$ and so

$$I = \cup_{k\in\mathbb{Z}}(\pi/2 + 2k\pi, 3\pi/2 + 2k\pi).$$

Therefore, in order to find information about t_c, we would need to optimise the function $h(s) = -\cos(s)^{-1}$ on the interval $(\pi/2, 3\pi/2)$ (notice that this function is 2π-periodic, so it is enough to consider this interval). We have clearly

$$\lim_{s\downarrow\pi/2} h(s) = \lim_{s\uparrow 3\pi/2} h(s) = +\infty. \tag{3}$$

Since $h(\pi) = 1$, we certainly have that $t_c \le 1$. In fact, as the maximum of $|\cos(s)|$ is 1, $t_c = 1$.

3. In this problem we consider harmonic function on the unit ball in $\mathbb{R}^3$, $B_1(0)$.

   (a) Using the fact that the Laplacian in spherical coordinates is given by

$$\Delta\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

   show that if $u \in C^2\left(\overline{B_1(0)}\right)$ is radial (i.e. only depends on r in spherical coordinates) and harmonic in $B_1(0)$ then it must be constant.

   (b) Show that there exists no radial solution in $C^2\left(\overline{B_1(0)}\right)$ to the equation

$$\begin{cases} -\Delta u(\vec{x}) = 0, & \vec{x} \in B_1(0), \\ u(\vec{x}) = f(\vec{x}), & \vec{x} \in \partial B_1(0), \end{cases}$$

   for $f(\vec{x}) = x_1^2$.

Solution:

3.(a) Given a harmonic function $u(r)$ that solves the equation $\Delta u(x) = 0$ we find that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2u'(r)\right) = 0$$

which implies that

$$r^2u'(r) = C$$

or

$$u'(r) = \frac{C}{r^2}.$$

Integration gives us that the general radial solution for Laplace's equation is

$$u(r) = -\frac{C}{r} + B.$$

As u needs to be continuous we find that the only harmonic radial function in $B_1(0)$ is a constant function.

3.(b) Since the boundary condition is not a constant function there can be no radial harmonic function which solves the equation.

4. Consider the heat-like equation

$$\begin{cases} u_t - u_{xx} + cu = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{4}$$

where $c \in \mathbb{R}$ is a fixed constant and $g \in C_c(\mathbb{R})$.

(a) Define $v(x,t) = e^{ct}u(x,t)$. Show that $v(x,t)$ solves the heat equation

$$\begin{cases} v_t - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ v(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(b) Show that there exists a solution to (4) that satisfies

$$\sup_{x \in \mathbb{R}} |u(x,t)| \leq e^{-ct} \|g\|_{L^\infty(\mathbb{R})}.$$

You may use the following inequality without proof: For any $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ we have that

$$\left| \int_{\mathbb{R}} f(x-y)\, g(y) \right| \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})}, \qquad \forall x \in \mathbb{R}.$$

Solution:

4.(a) We have that

$$v_t - v_x x = e^{ct}u_t + ce^{ct}u - e^{ct}u_{xx} = e^{ct}(u_t - u_{xx} + cu) = 0.$$

Additionally

$$v(x,0) = e^0 u(x,0) = g(x).$$

4.(b) From class we know that $v(x,t) = (\Phi(\cdot,t) * g)(x)$ is a solution to the equation.
Moreover, Φ is non-negative and $\int_{\mathbb{R}} \Phi(x,t)dx = 1$. Using the given inequality we find that

$$|v(t,x)| = |\Phi * g(x,t)| \leq \|\Phi\|_{L^1(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} = \|g\|_{L^\infty(\mathbb{R})}$$

from which we conclude that for any $x \in \mathbb{R}$ the solution u that is associated to the above v satisfies

$$|u(x,t)| = e^{-ct} |v(x,t)| \leq e^{-ct} \|g\|_{L^\infty(\mathbb{R})},$$

which shows the desired result.

Section B

5. We consider the following conservation law

$$\begin{cases} \partial_t u(x,t) - u(x,t)\partial_x u(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

[Notice that this is *not* Burgers' equation.]

(a) Suppose that u_0 is bounded, differentiable with bounded derivative. Give a formula of the critical time t_c, for which we know that (5) has a classical solution on $\mathbb{R} \times (0, t_c)$.

(b) Let $u_0(x) = -\arctan(x)$. Show that in this case (5) has a global in time classical solution.

(c) Let u_0 now be given by

$$u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

By drawing the characteristics, show that there is instantaneous crossing of characteristics. Find a shock that satisfies the Rankine–Hugoniot condition. Give the expression of the weak solution in this case.

Solution:

5.(a) We notice here that the flux function is $f(s) = -\frac{1}{2}s^2$ and the wave speed is $c(s) = -s$. From the lectures we know that for $s \in \mathbb{R}$, the characteristics in the (x,t)-half-plane are given by

$$x = s + c(u_0(s))t,$$

and the critical time is defined as

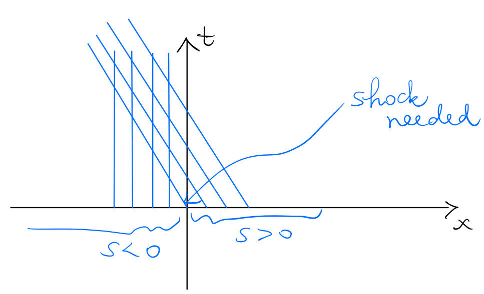
$$t_c := \inf\left\{ -\frac{1}{\partial_s(c(u_0(s)))} = \frac{1}{u_0'(s)} : s \in I \right\},$$

where $I = \{s \in \mathbb{R} : \partial_s(c(u_0(s))) < 0\} = \{s \in \mathbb{R} : u_0'(s) > 0\}$.

5.(b) As $u_0'(s) = -\frac{1}{1+s^2} < 0$, we have that $I = \emptyset$, and by definition $t_c = +\infty$.

5.(c) We have that the characteristics are given by

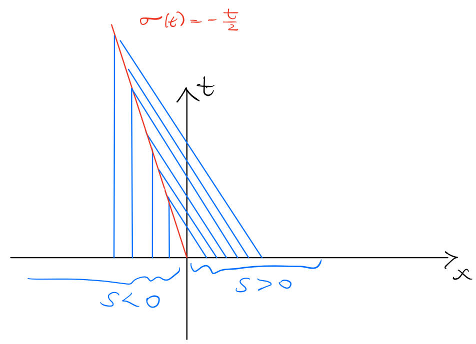
$$x = s, \quad \text{if } s < 0; \quad \text{and} \quad x = s - t, \text{ if } s \geq 0.$$



To introduce a shock that satisfies the Rankine–Hugoniot condition, this must satisfy $\sigma(0) = 0$ and

$$(u_\ell - u_r)\dot{\sigma} = f(u_\ell) - f(u_r) = -\frac{1}{2}(u_\ell - u_r)(u_\ell + u_r),$$

from where $\dot{\sigma} = -\frac{1}{2}$ and so, $\sigma(t) = -\frac{t}{2}$. The figure below includes this shock.



Therefore, the weak integral solution in this case writes as

$$u(x,t) = \begin{cases} 0, & x < -t/2, \\ 1, & x > t/2. \end{cases}$$

6. We aim to solve the following problem by the method of characteristics

$$\begin{cases} \partial^2_{xx}u - 3\partial^2_{xy}u + 2\partial^2_{yy}u = 0, & (x,y) \in \mathbb{R}^2, \\ u(1,y) = g(y), & y \in \mathbb{R}, \\ \partial_x u(1,y) = h(y), & y \in \mathbb{R}, \end{cases} \tag{6}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are given smooth functions.

(a) Identify the Cauchy data and the Cauchy curve in the above problem.

(b) Rewrite the PDE in (6) as a system of two linear first order PDEs. [*Hint*: think about the algebraic relation $(a-b)(a-2b) = a^2 - 3ab + 2b^2,\ (a, b \in \mathbb{R})$.]

(c) By solving the two first order PDEs arising from 6b using the method of characteristics, find the solution to (6).

Solution:

6.(a) The Cauchy data are the functions g and h, while the Cauchy curve is given by $\{\gamma(s) = (1, s) : s \in \mathbb{R}\} \subset \mathbb{R}^2$.

6.(b) Using the hint, we can write

$$\partial^2_{xx}u - 3\partial^2_{xy}u + 2\partial^2_{yy}u = (\partial_x - \partial_y)(\partial_x - 2\partial_y)u = 0.$$

Therefore, introduce v such that

$$\begin{cases} \partial_x u - 2\partial_y u = v, \\ \partial_x v - \partial_y v = 0. \end{cases}$$

The corresponding boundary conditions read as

$$\begin{cases} v(1,y) = h(y) - 2g'(y), & y \in \mathbb{R}, \\ u(1,y) = g(y), & y \in \mathbb{R}. \end{cases}$$

6.(c) We first solve the equation for v. Using the method of characteristics, we can write the ODE system

$$\begin{cases} \partial_\tau X(\tau, s) = 1, \\ \partial_\tau Y(\tau, s) = -1, \\ \partial_\tau z(\tau, s) = 0. \end{cases} \qquad \begin{cases} X(0, s) = 1, \\ Y(0, s) = s, \\ z(0, s) = h(s) - 2g'(s). \end{cases}$$

Solving this system, we obtain

$$\begin{cases} X(\tau, s) = \tau + 1, \\ Y(\tau, s) = -\tau + s, \\ z(\tau, s) = h(s) - 2g'(s). \end{cases}$$

For a given arbitrary point $(x, y) \in \mathbb{R}^2$, by inverting the flow map, we find

$$\begin{cases} \tau + 1 = x, \\ -\tau + s = y, \end{cases} \quad \text{and so} \quad \begin{cases} \tau = x - 1, \\ s = y + x - 1, \end{cases}$$

from where the solution reads as

$$v(x, y) = h(y + x - 1) - 2g'(y + x - 1).$$

Now we focus on solving the PDE for u, for this we write the ODE system

$$\begin{cases} \partial_\tau X(\tau,s) = 1, \\ \partial_\tau Y(\tau,s) = -2, \\ \partial_\tau z(\tau,s) = v(X(\tau,s),Y(\tau,s)), \end{cases} \qquad \begin{cases} X(0,s) = 1, \\ Y(0,s) = s, \\ z(0,s) = g(s). \end{cases}$$

Solving this system, we obtain

$$\begin{cases} X(\tau,s) = \tau + 1, \\ Y(\tau,s) = -2\tau + s, \\ z(\tau,s) = g(s) + \int_0^\tau v(X(t,s),Y(t,s))dt = \int_0^\tau \left(h(s-t) - 2g'(s-t)\right) dt. \end{cases}$$

For a given arbitrary point $(x,y) \in \mathbb{R}^2$, by inverting the flow map, we find

$$\begin{cases} \tau + 1 = x, \\ -2\tau + s = y, \end{cases} \quad \text{and so} \quad \begin{cases} \tau = x - 1, \\ s = y + 2x - 2, \end{cases}$$

therefore, the general solution for u, and hence of the original problem reads as

$$\begin{aligned} u(x,y) &= g(y+2x-2) + \int_0^{x-1} \big(h(y+2x-2-t) - 2g'(y+2x-2-t)\big)\, dt \\ &= \int_0^{x-1} h(y+2x-2-t)dt + 2g(y+x-1) - g(y+2x-2). \end{aligned}$$

From this expression we see

$$\begin{aligned} u(1,y) &= g(y), \\ \partial_x u(1,y) &= h(y), \end{aligned}$$

and by direct computation, we can also check that this u satisfies also the PDE.

7. Let Ω be an open bounded set with smooth boundary in $\mathbb{R}^n$ and let u_1 and u_2 be $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solutions to Poisson equation

$$\begin{cases} -\Delta u_i(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u_i(\vec{x}) = g_i(\vec{x}), & \vec{x} \in \partial\Omega, \end{cases}$$

$i = 1, 2$, where $f \in C^1\left(\overline{\Omega}\right)$ and $g_1, g_2 \in C(\partial\Omega)$.

(a) Show that for any $\vec{x} \in \overline{\Omega}$

$$u_2(\vec{x}) - u_1(\vec{x}) \leq \max_{\vec{x} \in \partial\Omega} \left(g_2(\vec{x}) - g_1(\vec{x})\right).$$

(b) Show that

$$\max_{\vec{x} \in \overline{\Omega}} |u_2(\vec{x}) - u_1(\vec{x})| \leq \max_{\vec{x} \in \partial\Omega} |g_2(\vec{x}) - g_1(\vec{x})|.$$

(c) For $n \in \mathbb{N}$ let $u_n \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solve the system

$$\begin{cases} -\Delta u_n(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u_n(\vec{x}) = g_n(\vec{x}), & \vec{x} \in \partial\Omega, \end{cases}$$

and let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ solve the system

$$\begin{cases} -\Delta u(\vec{x}) = f(\vec{x}), & \vec{x} \in \Omega, \\ u(\vec{x}) = g(\vec{x}), & \vec{x} \in \partial\Omega. \end{cases}$$

Show that if $\{g_n\}_{n \in \mathbb{N}}$ converges uniformly to g on $\partial\Omega$ then $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to u on $\overline{\Omega}$.

Recall that we say that a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ in $C(K)$ converges uniformly to $f \in C(K)$ if

$$\sup_{x \in K} |f_n(x) - f(x)| \underset{n \to \infty}{\longrightarrow} 0.$$

Solution:

7.(a) Since $w = u_2 - u_1$ solves Laplace's equation

$$\begin{aligned} -\Delta w(\vec{x}) &= 0 & \vec{x} \in \Omega \\ w(\vec{x}) &= g_2(\vec{x}) - g_1(\vec{x}) & \vec{x} \in \partial\Omega \end{aligned}$$

we can use the weak maximum principle to conclude that

$$\max_{\overline{\Omega}} w(\vec{x}) \leq \max_{\vec{x} \in \partial\Omega} \left(g_2(\vec{x}) - g_1(\vec{x})\right),$$

from which we conclude that for any $\vec{x} \in \overline{\Omega}$

$$u_2(\vec{x}) - u_1(\vec{x}) \leq \max_{\vec{x} \in \partial\Omega} \left(g_2(\vec{x}) - g_1(\vec{x})\right).$$

7.(b) Interchanging u_1 with u_2 we find that for any $\vec{x} \in \overline{\Omega}$

$$u_1(\vec{x}) - u_2(\vec{x}) \leq \max_{\vec{x} \in \partial\Omega} \left(g_1(\vec{x}) - g_2(\vec{x})\right).$$

Consequently

$$\begin{aligned} \max_{\vec{x} \in \overline{\Omega}} |u_2(\vec{x}) - u_1(\vec{x})| &= \max_{\vec{x} \in \overline{\Omega}} \max\left(u_2(\vec{x}) - u_1(\vec{x}), u_1(\vec{x}) - u_2(\vec{x})\right) \\ &\leq \max_{\vec{x} \in \Omega} \max\left(\max_{\vec{x} \in \partial\Omega} \left(g_2(\vec{x}) - g_1(\vec{x})\right), \max_{\vec{x} \in \partial\Omega} \left(g_1(\vec{x}) - g_2(\vec{x})\right)\right) \\ &\leq \max_{x \in \partial\Omega} |g_1(\vec{x}) - g_2(\vec{x})|. \end{aligned}$$

7.(c) Using our previous estimation we find that

$$\max_{\vec{x}\in\overline{\Omega}} |u(\vec{x}) - u_n(\vec{x})| \leq \max_{\vec{x}\in\partial\Omega} |g(\vec{x}) - g_n(\vec{x})| \underset{n\to\infty}{\longrightarrow} 0.$$

This concludes the proof.

8. Let $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a classical solution to the equation

$$\begin{cases} u_t + k u_{xxxx} = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $k > 0$ is a fixed constant and f is a smooth function on $\mathbb{R}$ that belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(a) Show that $\widehat{u}$, the Fourier transform of u in the $x-$variable, satisfies

$$\widehat{u}(\xi,t) = \widehat{f}(\xi) e^{-k\xi^4 t}.$$

(b) Using the fact that the Fourier transform preserves the L^2 norm (Plancherel's identity) as well as the fact that it is in L^∞ to show that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} \leq \frac{\left(\int_{\mathbb{R}} e^{-x^4} dx\right)^{\frac{1}{2}}}{\sqrt[8]{4kt}} \|f\|_{L^1(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

Solution:

8.(a) Using the linearity of the Fourier transform together with the identity

$$\widehat{u'}(\xi) = i\xi \widehat{u}(\xi)$$

we find that using the Fourier transform on the spatial variable in our equation yields the equation

$$\widehat{u}_t(\xi,t) + k(i\xi)^4 \widehat{u}(\xi,t) = 0.$$

This is a separable equation whose solution is

$$\widehat{u}(\xi,t) = e^{-k\xi^4 t} \widehat{u}(\xi,0) = e^{-k\xi^4 t} \widehat{f}(\xi)$$

since $u(x,0) = f(x)$.

8.(b) Using the fact that for any $f \in L^1(\mathbb{R})$

$$\left\|\widehat{f}\right\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$$

together with Plancherel identity and Cauchy-Schwarz we find that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|\widehat{u}(\cdot,t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|e^{-k\xi^4 t} \widehat{f}(\xi)\right|^2 d\xi$$

$$\leq \left\|\widehat{f}\right\|_{L^\infty(\mathbb{R})} \left(\int_{\mathbb{R}} e^{-2k\xi^4 t} \left|\widehat{f}(\xi)\right| d\xi\right) \leq \|f\|_{L^1(\mathbb{R})} \left\|\widehat{f}\right\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} e^{-4k\xi^4 t} d\xi\right)^{\frac{1}{2}}$$

$$\underset{x = \sqrt[4]{4kt}\xi}{=} \frac{\left(\int_{\mathbb{R}} e^{-x^4} dx\right)^{\frac{1}{2}}}{\sqrt[8]{4kt}} \|f\|_{L^1(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$