PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 5

Exercise 1 *(Hölder inequality)*. The so-called Hölder inequality which states that for for (measurable) set $E \subseteq \mathbb{R}^n$ and any $f \in L^p(E)$ and $g \in L^q(E)$ where p and q Hölder conjugate, i.e. $p, q \in [1, \infty]$ and $1/p + 1/q = 1$ we have that[1]

$$\int_E |f(x)g(x)|\, dx \leq \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

where $L^p(E)$ is defined like $L^p(\mathbb{R}^n)$ where the set on which we integrate is E instead of $\mathbb{R}^n$.

(i) Prove Hölder inequality when $E = \mathbb{R}^n$, $p = \infty$ and $q = 1$. You may assume that f is bounded on $\mathbb{R}^n$ (and as such use sup instead of esssup).

(ii) Show that if $f \in C_c(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ then there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq \|f\|_{L^\infty} |K|^{\frac{1}{q}} \left(\int_K |g(x)|^p\, dx \right)^{\frac{1}{p}}.$$

(iii) Use Hölder inequality to show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with p and q Hölder conjugates we have that

$$\sup_{x \in \mathbb{R}^n} |f * g(x)| = \|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

[1] One particular case that is worth to mention is when $p = q = 2$. This case is the famous Cauchy-Schwarz inequality and it reads as

$$\int_E |f(x)g(x)|\, dx \leq \left(\int_E |f(x)|^2\, dx \right)^{\frac{1}{2}} \left(\int_E |g(x)|^2\, dx \right)^{\frac{1}{2}}$$

Exercise 2 *(Poincaré-like inequalities).*

(i) Using the formula

$$f(x) - f(y) = \int_y^x f'(s)\,ds,$$

which holds for any C^1 function on an interval that contains x and y, together with Hölder's inequality show that for any Hölder conjugates p and q, any $f \in C^1([a,b])$, and any $c \in [a,b]$ we have that

$$\sup_{x\in[a,b]} \left|f(x) - f(c)\right| \leq (b-a)^{\frac{1}{q}} \left\|f'\right\|_{L^p([a,b])},$$

where q is the Hölder conjugate of p.

(ii) Conclude that for any $p \in [1,\infty]$ and $r \in [1,\infty)$, any $f \in C^1([a,b])$, and any $c \in [a,b]$ we have that

$$\left(\int_a^b \left|f(x) - f(c)\right|^r dx\right)^{\frac{1}{r}} \leq (b-a)^{\frac{1}{q}+\frac{1}{r}} \left\|f'\right\|_{L^p([a,b])}.$$

or equivalently

$$\left\|f - f(c)\right\|_{L^r([a,b])} \leq (b-a)^{\frac{1}{q}+\frac{1}{r}} \left\|f'\right\|_{L^p([a,b])}.$$

Note that due to our previous sub-question the above remains true when $r = \infty$.

(iii) Use sub-question ii to conclude that for any $f \in C^1([a,b])$ and any $p \in [1,\infty]$ we have

$$\left\|f - \overline{f}\right\|_{L^p([a,b])} \leq (b-a) \left\|f'\right\|_{L^p([a,b])}$$

where

$$\overline{f} = \frac{1}{b-a}\int_a^b f(s)\,ds.$$

(iv) Use sub-question ii to conclude that for any $f \in C^1([a,b])$ such that $f(c) = 0$ for some $c \in [a,b]$ and any $p \in [1,\infty]$ we have

$$\left\|f\right\|_{L^p([a,b])} \leq (b-a) \left\|f'\right\|_{L^p([a,b])}.$$

Exercise 3 *(The Poincaré constant).*

(i) Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the PDE

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \quad \text{in } \Omega, \\ u(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $u \in C^2\left(\overline{\Omega}\right)$ and $\lambda \in \mathbb{R}$. By multiplying the above by u and integrating by parts show that if $u \neq 0$ we have that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega u^2(x)\, dx} \geq 0. \tag{2}$$

This is known as the *Rayleigh quotient formula* for the eigenvalue λ in terms of the eigenfunction u.

Remark: This procedure, where we multiplied by a function, integrated, and recovered a functional that helps us understand our equation better is known as the *energy method* (the functional we found is our "energy"). This is an important method which will repeat in this module.

Remark: We could allow u to have values in $\mathbb{C}$. In that case we would multiply our PDE with $\overline{u}$ and conclude that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega |u(x)|^2\, dx} \geq 0.$$

(ii) Show that any λ that satisfies our PDE must satisfy

$$\lambda \geq \frac{1}{C_P^2}$$

where C_P is the Poincaré inequality that is associated to Ω and the Dirichlet boundary condition, i.e. the best constant for which

$$\int_\Omega |u(x)|^2\, dx \leq C_P^2 \int_\Omega |\nabla u(x)|^2\, dx.$$

Remark: The above implies that

$$\min_{\text{eigenvalues}} \lambda \geq \frac{1}{C_P^2}.$$

One can in fact show that there is equality in the above. This is connected to minimising the energy

$$E[u] = \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega |v(x)|^2\,dx}$$

over an appropriate space of *functions*.