PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 5

Exercise 1 *(Hölder inequality)*. The so-called Hölder inequality which states that for for (measurable) set $E \subseteq \mathbb{R}^n$ and any $f \in L^p(E)$ and $g \in L^q(E)$ where p and q Hölder conjugate, i.e. $p, q \in [1, \infty]$ and $1/p + 1/q = 1$ we have that[1]

$$\int_E |f(x)g(x)|\, dx \le \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

where $L^p(E)$ is defined like $L^p(\mathbb{R}^n)$ where the set on which we integrate is E instead of $\mathbb{R}^n$.

(i) Prove Hölder inequality when $E = \mathbb{R}^n$, $p = \infty$ and $q = 1$. You may assume that f is bounded on $\mathbb{R}^n$ (and as such use sup instead of esssup).

(ii) Show that if $f \in C_c(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ then there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le \|f\|_{L^\infty} |K|^{\frac{1}{q}} \left(\int_K |g(x)|^p\, dx\right)^{\frac{1}{p}}.$$

(iii) Use Hölder inequality to show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with p and q Hölder conjugates we have that

$$\sup_{x \in \mathbb{R}^n} |f * g(x)| = \|f * g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Sol:

(i)

$$\int_{\mathbb{R}^n} |f(x) g(x)|\, dx = \int_{\mathbb{R}^n} |f(x)|\,|g(x)|\, dx$$

$$\le \int \underbrace{\|f\|_{L^\infty(\mathbb{R}^n)}}_{\sup_{x\in\mathbb{R}^n}|f(x)|} |g(x)|\, dx = \|f\|_{L^\infty(\mathbb{R}^n)} \underbrace{\int_{\mathbb{R}^n} |g(x)|\, dx}_{\|g\|_{L^1(\mathbb{R}^n)}}$$

[1] One particular case that is worth to mention is when $p = q = 2$. This case is the famous Cauchy-Schwarz inequality and it reads as

$$\int_E |f(x)g(x)|\, dx \le \left(\int_E |f(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_E |g(x)|^2\, dx\right)^{\frac{1}{2}}$$

(ii) Since $f \in C_c(\mathbb{R}^n)$ $\exists K$ compact s.t $f|_{K^c} = 0$

$$\int_{\mathbb{R}^n} |f(x) g(x)|\,dx = \int_K |f(x) g(x)|\,dx \overset{\text{like (i)}}{\leq} \|f\|_{L^\infty(\mathbb{R}^n)} \int_K 1 \cdot |g(x)|\,dx$$

$$\overset{\text{Hölder}}{\leq} \|f\|_{L^\infty(\mathbb{R}^n)} \underbrace{\left(\int_K |1|^q\,dx\right)^{1/q}}_{|K|} \left(\int_K |g(x)|^p\,dx\right)^{1/p}$$

$$= \|f\|_{L^\infty(\mathbb{R}^n)} |K|^{1/q} \left(\int_K |g(x)|^p\,dx\right)^{1/p}.$$

<u>Note:</u> We can weaken the cond. to $g \in L^p_{loc}$

(iii) $$|f * g(x)| = \left|\int_{\mathbb{R}^n} f(x-y)\, g(y)\,dy\right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy \overset{\text{Hölder}}{\leq} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dy\right)^{1/p}$$

$$\cdot \left(\int_{\mathbb{R}^n} |g(y)|^q\,dy\right)^{1/q} \underset{\substack{z = x-y \\ dz = \underbrace{J}_{1} dy}}{=} \left(\int_{\mathbb{R}^n} |f(z)|^p\,dz\right)^{1/p} \left(\int_{\mathbb{R}^n} |g(y)|^q\,dy\right)^{1/q}$$

$\boxtimes$

Exercise 2 *(Poincaré-like inequalities).*

(i) Using the formula
$$f(x)-f(y)=\int_y^x f'(s)ds,$$
which holds for any C^1 function on an interval that contains x and y, together with Hölder's inequality show that for any Hölder conjugates p and q, any $f\in C^1([a,b])$, and any $c\in[a,b]$ we have that
$$\sup_{x\in[a,b]}\left|f(x)-f(c)\right|\leq(b-a)^{\frac{1}{q}}\left\|f'\right\|_{L^p([a,b])},$$
where q is the Hölder conjugate of p.

(ii) Conclude that for any $p\in[1,\infty]$ and $r\in[1,\infty)$, any $f\in C^1([a,b])$, and any $c\in[a,b]$ we have that
$$\left(\int_a^b\left|f(x)-f(c)\right|^r dx\right)^{\frac{1}{r}}\leq(b-a)^{\frac{1}{q}+\frac{1}{r}}\left\|f'\right\|_{L^p([a,b])}.$$
or equivalently
$$\left\|f-f(c)\right\|_{L^r([a,b])}\leq(b-a)^{\frac{1}{q}+\frac{1}{r}}\left\|f'\right\|_{L^p([a,b])}.$$
Note that due to our previous sub-question the above remains true when $r=\infty$.

(iii) Use sub-question ii to conclude that for any $f\in C^1([a,b])$ and any $p\in[1,\infty]$ we have
$$\left\|f-\overline{f}\right\|_{L^p([a,b])}\leq(b-a)\left\|f'\right\|_{L^p([a,b])}$$
where
$$\overline{f}=\frac{1}{b-a}\int_a^b f(s)ds.$$

(iv) Use sub-question ii to conclude that for any $f\in C^1([a,b])$ such that $f(c)=0$ for some $c\in[a,b]$ and any $p\in[1,\infty]$ we have
$$\|f\|_{L^p([a,b])}\leq(b-a)\left\|f'\right\|_{L^p([a,b])}.$$

Sol:

(i) for any $c\in[a,b]$ and $x\in[a,b]$

$$|f(x)-f(c)|=\left|\int_c^x f'(s)ds\right|\leq\int_c^x 1\cdot|f'(s)|ds$$

$$\underset{\text{Hölder}}{\leq}\left(\int_c^x 1^q ds\right)^{1/q}\left(\int_c^x|f'(s)|^p ds\right)^{1/p}$$

$$\leq\left(\int_a^b 1ds\right)^{1/q}\left(\int_a^b|f'(s)|^p ds\right)^{1/p}=(b-a)^{1/q}\|f'\|_{L^p([a,b])}$$

$\Rightarrow \forall c \in [a,b]$

$$\underbrace{\sup_{x\in[a,b]} |f(x)-f(c)|}_{\|f-f(c)\|_{L^\infty([a,b])}} \le (b-a)^{1/q} \|f'\|_{L^p([a,b])}$$

(ii) $$\left(\int_a^b |f(x)-f(c)|^r \, dx\right)^{1/r} \le \left(\int_a^b \left(\sup_x |f(x)-f(c)|\right)^r dx\right)^{1/r}$$

$$= \underbrace{\sup_{x\in[a,b]} |f(x)-f(c)|}_{\le (b-a)^{1/q}\|f'\|_{L^p([a,b])}} \underbrace{\left(\int_a^b 1 \, dx\right)^{1/r}}_{(b-a)^{1/r}}$$

$$\le (b-a)^{1/q+1/r} \|f'\|_{L^p([a,b])}$$

(iii) Choosing $r=p$ $\left(\frac{1}{q}+\frac{1}{p}=1\right)$ and $c\in[a,b]$ s.t. $f(c)=\bar{f}$ (integral IVT) in (ii) gives the result.

(iv) Choosing the c s.t. $f(c)=0$ and $r=p$ in (ii) gives the result

Exercise 3 *(The Poincaré constant).*

(i) Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the PDE

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \quad \text{in } \Omega, \\ u(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $u \in C^2\left(\overline{\Omega}\right)$ and $\lambda \in \mathbb{R}$. By multiplying the above by u and integrating by parts show that if $u \neq 0$ we have that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega u^2(x) \, dx} \geq 0. \tag{2}$$

This is known as the *Rayleigh quotient formula* for the eigenvalue λ in terms of the eigenfunction u.

Remark: This procedure, where we multiplied by a function, integrated, and recovered a functional that helps us understand our equation better is known as the *energy method* (the functional we found is our "energy"). This is an important method which will repeat in this module.

Remark: We could allow u to have values in $\mathbb{C}$. In that case we would multiply our PDE with $\overline{u}$ and conclude that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega |u(x)|^2 \, dx} \geq 0.$$

(ii) Show that any λ that satisfies our PDE must satisfy

$$\lambda \geq \frac{1}{C_P^2}$$

where C_P is the Poincaré inequality that is associated to Ω and the Dirichlet boundary condition, i.e. the best constant for which

$$\int_\Omega |u(x)|^2 \, dx \leq C_P^2 \int_\Omega |\nabla u(x)|^2 \, dx.$$

Remark: The above implies that

$$\min_{\text{eigenvalues}} \lambda \geq \frac{1}{C_P^2}.$$

One can in fact show that there is equality in the above. This is connected to minimising the energy

$$E[u] = \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega |v(x)|^2\,dx}$$

over an appropriate space of *functions*.

Sol:

(i) $-u\Delta u = \lambda u^2$

$$\Rightarrow \int_\Omega -u(x)\Delta u(x)\,dx = \lambda \int_\Omega u(x)^2\,dx$$

$$u\Delta u = u \cdot \operatorname{div}(\vec\nabla u) = \operatorname{div}(u\vec\nabla u) - \underbrace{\vec\nabla u \cdot \vec\nabla u}_{|\vec\nabla u|^2}$$

$$\Rightarrow \int_\Omega -u(x)\Delta u(x)\,dx = \int_\Omega \left[|\nabla u(x)|^2 dx - \operatorname{div}(u\vec\nabla u(x))\right]dx$$

$u|_{\partial\Omega} = 0$

$$= \int_\Omega |\nabla u(x)|^2\,dx - \cancel{\int_{\partial\Omega} u(y)\vec\nabla u(y)\cdot \hat n(y)\,dS(y)}$$

0

$$\Rightarrow \int_\Omega |\nabla u(x)|^2\,dx = \lambda \int_\Omega u(x)^2\,dx$$

✓

(ii) see sol.