PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 5 SOLUTION

Exercise 1 *(Hölder inequality).* The so-called Hölder inequality which states that for for (measurable) set $E \subseteq \mathbb{R}^n$ and any $f \in L^p(E)$ and $g \in L^q(E)$ where p and q Hölder conjugate, i.e. $p, q \in [1, \infty]$ and $1/p + 1/q = 1$ we have that[1]

$$\int_E |f(x)g(x)|\,dx \le \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

where $L^p(E)$ is defined like $L^p(\mathbb{R}^n)$ where the set on which we integrate is E instead of $\mathbb{R}^n$.

(i) Prove Hölder inequality when $E = \mathbb{R}^n$, $p = \infty$ and $q = 1$. You may assume that f is bounded on $\mathbb{R}^n$ (and as such use sup instead of esssup).

(ii) Show that if $f \in C_c(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ then there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|f\|_{L^\infty} |K|^{\frac{1}{q}} \left(\int_K |g(x)|^p\,dx\right)^{\frac{1}{p}}.$$

(iii) Use Hölder inequality to show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with p and q Hölder conjugates we have that

$$\sup_{x\in\mathbb{R}^n} |f * g(x)| = \|f * g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Solution. (i) We have that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \int_{\mathbb{R}^n} \left(\sup_{x\in\mathbb{R}^n} |f(x)|\right) |g(x)|\,dx = \|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |g(x)|\,dx$$
$$= \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

which is the desired inequality.

(ii) Since $f \in C_c(\mathbb{R}^n)$ we know that there exists a compact set $K \subset \mathbb{R}^n$ such that $f|_{K^c} = 0$ and that $\|f\|_\infty < \infty$. We conclude that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx = \int_K |f(x)g(x)|\,dx \le \|f\|_{L^\infty} \int_K |g(x)|\,dx = \|f\|_{L^\infty} \int_K 1\ |g(x)|\,dx$$

[1] One particular case that is worth to mention is when $p = q = 2$. This case is the famous Cauchy-Schwarz inequality and it reads as

$$\int_E |f(x)g(x)|\,dx \le \left(\int_E |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_E |g(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq \|f\|_{L^\infty} \|1\|_{L^q(K)} \|g\|_{L^p(K)} = \|f\|_{L^\infty} |K|^{\frac{1}{q}} \left(\int_K |g(x)|^p dx\right)^{\frac{1}{p}}.$$

(iii) We have that

$$|f*g(x)| = \left|\int_{\mathbb{R}^n} f(x-y)g(y)dy\right| \leq \int_{\mathbb{R}^n} |f(x-y)g(y)dy| \leq \|f(x-\cdot)\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Since

$$\|f(x-\cdot)\|_{L^p(\mathbb{R}^n)} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x-y)|^p dy\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \mathrm{esssup}_{y\in\mathbb{R}^n} |f(x-y)|, & p = \infty, \end{cases}$$

$$= \begin{cases} \left(\int_{\mathbb{R}^n} |f(z)|^p dz\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \mathrm{esssup}_{z\in\mathbb{R}^n} |f(z)|, & p = \infty, \end{cases} = \|f\|_{L^p(\mathbb{R}^n)}$$

we conclude the desired result.

□

Exercise 2 *(Poincaré-like inequalities).*

(i) Using the formula

$$f(x) - f(y) = \int_y^x f'(s)ds,$$

which holds for any C^1 function on an interval that contains x and y, together with Hölder's inequality show that for any Hölder conjugates p and q, any $f \in C^1([a,b])$, and any $c \in [a,b]$ we have that

$$\sup_{x\in[a,b]} |f(x) - f(c)| \leq (b-a)^{\frac{1}{q}} \|f'\|_{L^p([a,b])},$$

where q is the Hölder conjugate of p.

(ii) Conclude that for any $p \in [1,\infty]$ and $r \in [1,\infty)$, any $f \in C^1([a,b])$, and any $c \in [a,b]$ we have that

$$\left(\int_a^b |f(x) - f(c)|^r dx\right)^{\frac{1}{r}} \leq (b-a)^{\frac{1}{q}+\frac{1}{r}} \|f'\|_{L^p([a,b])}.$$

or equivalently

$$\|f - f(c)\|_{L^r([a,b])} \leq (b-a)^{\frac{1}{q}+\frac{1}{r}} \|f'\|_{L^p([a,b])}.$$

Note that due to our previous sub-question the above remains true when $r = \infty$.

(iii) Use sub-question ii to conclude that for any $f \in C^1([a,b])$ and any $p \in [1,\infty]$ we have

$$\left\|f - \overline{f}\right\|_{L^p([a,b])} \leq (b-a) \|f'\|_{L^p([a,b])}$$

where
$$\overline{f} = \frac{1}{b-a}\int_a^b f(s)ds.$$

(iv) Use sub-question ii to conclude that for any $f \in C^1([a,b])$ such that $f(c) = 0$ for some $c \in [a,b]$ and any $p \in [1,\infty]$ we have
$$\|f\|_{L^p([a,b])} \le (b-a)\|f'\|_{L^p([a,b])}.$$

Solution.

We have
$$\left|f(x)-f(c)\right| = \left|\int_c^x f'(s)ds\right| \le \int_c^x \left|f'(s)\right| ds = \int_c^x 1\cdot\left|f'(s)\right| ds \le \int_a^b 1\cdot\left|f'(s)\right| ds.$$

Using Hölder inequality we find that
$$\left|f(x)-f(c)\right| \le \|1\|_{L^q([a,b])}\|f'\|_{L^p([a,b])} = (b-a)^{\frac{1}{q}}\|f'\|_{L^p([a,b])}.$$

(ii) Since
$$\int_a^b \left|f(x)-f(c)\right|^r dx \le \left(\sup_{x\in[a,b]}\left|f(x)-f(c)\right|\right)^r \int_a^b 1dx = (b-a)\left(\sup_{x\in[a,b]}\left|f(x)-f(c)\right|\right)^r$$

we conclude from the previous sub-question that
$$\left(\int_a^b \left|f(x)-f(c)\right|^r\right)^{\frac{1}{r}} dx \le (b-a)^{\frac{1}{r}} \sup_{x\in[a,b]}\left|f(x)-f(c)\right|$$
$$\le (b-a)^{\frac{1}{q}+\frac{1}{r}}\|f'\|_{L^p([a,b])}$$

which is the desired result.

(iii) The mean value theorem for integrals says that for any two continuous functions on $[a,b]$, f and g, we have that there exists $c \in [a,b]$ such that
$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Choosing $g \equiv 1$ we find that there exists $c_* \in [a,b]$ such that
$$f(c_*) = \frac{\int_a^b f(x)dx}{\int_a^b 1dx} = \frac{1}{b-a}\int_a^b f(x)dx = \overline{f}.$$

plugging that into sub-question ii with $r = p$ gives the desired result.

(iv) Similarly, choosing $c = a$ and $r = p$ in sub-question ii gives us the desired result.

□

Exercise 3 *(The Poincaré constant).*

(i) Let Ω be an open bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the PDE

$$-\Delta u(x) = \lambda u(x) \quad \text{in } \Omega, \qquad u(x) = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $u \in C^2\left(\overline{\Omega}\right)$ and $\lambda \in \mathbb{R}$. By multiplying the above by u and integrating by parts show that if $u \neq 0$ we have that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega u^2(x) \, dx} \geq 0. \tag{2}$$

This is known as the *Rayleigh quotient formula* for the eigenvalue λ in terms of the eigenfunction u.

Remark: This procedure, where we multiplied by a function, integrated, and recovered a functional that helps us understand our equation better is known as the *energy method* (the functional we found is our "energy"). This is an important method which will repeat in this module.

Remark: We could allow u to have values in $\mathbb{C}$. In that case we would multiply our PDE with $\overline{u}$ and conclude that

$$\lambda = \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega |u(x)|^2 \, dx} \geq 0.$$

(ii) Show that any λ that satisfies our PDE must satisfy

$$\lambda \geq \frac{1}{C_P^2}$$

where C_P is the Poincaré inequality that is associated to Ω and the Dirichlet boundary condition, i.e. the best constant for which

$$\int_\Omega |u(x)|^2 \, dx \leq C_P^2 \int_\Omega |\nabla u(x)|^2 \, dx.$$

Remark: The above implies that

$$\min_{\text{eigenvalues}} \lambda \geq \frac{1}{C_P^2}.$$

One can in fact show that there is equality in the above. This is connected to minimising the energy

$$E[u] = \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} |v(x)|^2 \, dx}$$

over an appropriate space of *functions*.

Solution.

(i) Multiplying our equation by u and integrating we find that

$$-\int_{\Omega} u(x) \Delta u(x) dx = \lambda \int_{\Omega} u^2(x) dx.$$

Since

$$\operatorname{div}(u(x)\nabla u(x)) = |\nabla u(x)|^2 + u(x)\Delta u(x)$$

we conclude that

$$-\int_{\Omega} u(x)\Delta u(x) dx = \int_{\Omega} |\nabla u(x)|^2 \, dx - \int_{\Omega} \operatorname{div}(u(x)\nabla u(x)) \, dx$$

$$= \int_{\Omega} |\nabla u(x)|^2 \, dx - \int_{\partial\Omega} u(y)\nabla u(y) \cdot n(y) dS(y) = \int_{\Omega} |\nabla u(x)|^2 \, dx$$

where we have used the fact that $u|_{\partial\Omega} = 0$. We conclude that

$$\int_{\Omega} |\nabla u(x)|^2 \, dx = \lambda \int_{\Omega} u^2(x) dx$$

which is the desired result.

(ii) We know that any u which is C^1 and is zero on the boundary satisfies

$$\int_{\Omega} |u(x)|^2 \, dx \leq C_P^2 \int_{\Omega} |\nabla u(x)|^2 \, dx$$

and as such

$$\frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} |u(x)|^2} \geq \frac{1}{C_P^2}.$$

This implies that if u solves our PDE and $u \neq 0$ then

$$\lambda = \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} |u(x)|^2} \geq \frac{1}{C_P^2}.$$

□