PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 6

In all our exercises in this sheet we will assume that Ω is an open, bounded, and connected set with smooth boundary.

Exercise 1 *(Lower bound for Dirichlet's Energy)*. In this problem we will consider the Dirichlet Energy associated to the PDE

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

$$E[u] = \frac{1}{2}\int_{\Omega} |\nabla u(x)|^2 \, dx - \int_{\Omega} f(x) u(x) dx.$$

(i) Young's inequality states that for any $a, b \in \mathbb{R}$ and any Hölder conjugate numbers $p, q \in (1, \infty)$ we have that

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Consequently, for any $\varepsilon > 0$ we find that by replacing a with $(p\varepsilon)^{\frac{1}{p}} a$ and b with $\frac{b}{(p\varepsilon)^{\frac{1}{p}}}$ we get that

$$|ab| \le \varepsilon |a|^p + \frac{|b|^q}{q p^{q-1} \varepsilon^{q-1}}$$

and in particular that for any $\varepsilon > 0$ choosing $p = q = 2$ yields

$$|ab| \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Show that for any $\varepsilon > 0$ and any

$$u \in V = \left\{ v \in C^1\left(\overline{\Omega}\right) \mid v = 0 \text{ on } \partial\Omega \right\}$$

we have that

$$E[u] \ge \left(\frac{1}{2} - \varepsilon C_P(\Omega)^2\right) \|u\|^2_{H^1_0(\Omega)} - \frac{1}{4\varepsilon} \|f\|^2_{L^2(\Omega)}$$

where $C_P(\Omega)$ is the Poincaré constant associated to the domain Ω.

(ii) Conclude that there exists a constant $C > 0$ such that

$$\inf_{u \in V} E[u] \ge -C.$$

Exercise 2 *(Uniqueness of weak solutions)*. Show that if u and v are weak solutions, in the sense defined in class, for

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

then $u = v$.

Exercise 3 *(Uniqueness for a more general elliptic problem)*. Consider the linear, second-order, elliptic PDE

$$\begin{aligned} -\operatorname{div}(A\nabla u) + \mathbf{b}\cdot\nabla u + cu = f \quad & \text{in } \Omega, \\ u = g \quad & \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary, $A \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$, $\mathbf{b} \in C^1(\overline{\Omega};\mathbb{R}^n)$, and $c, f, g \in C(\overline{\Omega})$. Assume that c is non-negatives, $\operatorname{div}\mathbf{b} = 0$, and A is uniformly positive definite, i.e., there exists a constant $\alpha > 0$ such that $y^T A(x) y \geq \alpha|y|^2$ for all $y \in \mathbb{R}^n$, $x \in \Omega$. Prove that (2) has at most one solution $u \in C^2(\overline{\Omega})$.

Exercise 4. Consider the space

$$V = \{\varphi \in C^1(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega,\ \varphi \neq 0\}$$

and the functional $E : V \to \mathbb{R}$ defined by

$$E[v] = \frac{\displaystyle\int_\Omega |\nabla v(x)|^2\,dx}{\displaystyle\int_\Omega |v(x)|^2\,dx}.$$

Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E and show that

$$\begin{aligned} -\Delta u = \lambda u \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

where $\lambda = E[u]$.