PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 6

In all our exercises in this sheet we will assume that Ω is an open, bounded, and connected set with smooth boundary.

Exercise 1 *(Lower bound for Dirichlet's Energy).* In this problem we will consider the Dirichlet Energy associated to the PDE

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega f(x)u(x)\,dx.$$

(i) Young's inequality states that for any $a, b \in \mathbb{R}$ and any Hölder conjugate numbers $p, q \in (1,\infty)$ we have that

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Consequently, for any $\varepsilon > 0$ we find that by replacing a with $(p\varepsilon)^{\frac{1}{p}} a$ and b with $\frac{b}{(p\varepsilon)^{\frac{1}{p}}}$ we get that

$$|ab| \leq \varepsilon|a|^p + \frac{|b|^q}{qp^{q-1}\varepsilon^{q-1}}$$

and in particular that for any $\varepsilon > 0$ choosing $p = q = 2$ yields

$$|ab| \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Show that for any $\varepsilon > 0$ and any

$$u \in V = \left\{ v \in C^1\left(\overline{\Omega}\right) \mid v = 0 \text{ on } \partial\Omega \right\}$$

we have that

$$E[u] \geq \left(\frac{1}{2} - \varepsilon \overset{2}{C_P}(\Omega)\right)\|u\|^2_{H^1_0(\Omega)} - \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}$$

where $C_P(\Omega)$ is the Poincaré constant associated to the domain Ω.

(ii) Conclude that there exists a constant $C > 0$ such that

$$\inf_{u \in V} E[u] \geq -C.$$

__Sol:__

$$E[u] = \frac{1}{2}\underbrace{\int_\Omega |\nabla u(x)|^2\,dx}_{\|u\|^2_{H^1_0}} - \int_\Omega f(x)u(x)\,dx$$

$$\int_\Omega f(x)u(x)dx \le \int_\Omega |f(x)u(x)|dx \le$$

on $\mathbb{R}$

Young's ineq.

$$\int_\Omega \left(\varepsilon |u(x)|^2 + \frac{1}{4\varepsilon}|f(x)|^2\right)dx = \varepsilon \|u\|_{L^2}^2 + \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

Thus

$$E[u] \geq \frac{1}{2}\|u\|_{H_0^1}^2 - \varepsilon\|u\|_{L^2}^2 - \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

$$\underset{\text{Poincaré}}{\geq} \frac{1}{2}\|u\|_{H_0^1}^2 - \varepsilon C_p^2(\Omega)\|u\|_{H_0^1}^2 - \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

$u|_{\partial\Omega} = 0$

$\|u\|_{L^2} \le C_p(\Omega)\|u\|_{H_0^1}$

$$= \left(\frac{1}{2} - \varepsilon C_p^2(\Omega)\right)\|u\|_{H_0^1}^2 - \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

(ii) Choosing $\varepsilon > 0$ s.t. $\frac{1}{2} - \varepsilon C_p^2(\Omega) \geq 0$

we find that for $\underline{\text{any}}$ $u \in V$

$$E[u] \geq \underbrace{\left(\frac{1}{2} - \varepsilon C_p^2(\Omega)\right)}_{\geq 0}\underbrace{\|u\|_{H_0^1}^2}_{\geq 0} - \frac{1}{4\varepsilon}\|f\|_{L^2}^2$$

$$\geq -\frac{1}{4\varepsilon}\|f\|_{L^2}^2 = -C$$

for instance $\varepsilon = \frac{1}{2C_p^2(\Omega)}$ i.e. $E[u] \geq -\frac{C_p^2(\Omega)}{2}\|f\|_{L^2}^2$

Exercise 2 *(Uniqueness of weak solutions)*. Show that if u and v are weak solutions, in the sense defined in class, for

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

then $u = v$.

Sol: As u, v are weak sol. for any $\varphi \in V = \{\xi \in C^1(\bar{\Omega}) \mid \xi|_{\partial\Omega} = 0\}$

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx$$

$$\int_\Omega \nabla v(x) \cdot \nabla \varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx$$

Subtracting we get

$$\int_\Omega \nabla(u(x) - v(x)) \cdot \nabla \varphi(x)\,dx = 0$$

$\varphi = u - v \in V \Rightarrow$

$$\int_\Omega \nabla(u(x) - v(x)) \cdot \nabla(u(x) - v(x))\,dx = 0$$

i.e.

$$\int_\Omega \underbrace{|\nabla u(x) - \nabla v(x)|^2}_{\text{Cont. and non-negative}}\,dx = 0$$

$\Rightarrow |\nabla u(x) - \nabla v(x)|^2 = 0 \Rightarrow \nabla(u - v) = 0$

Ω is open and connected $\Rightarrow u - v = \text{Const}$

as $u - v|_{\partial\Omega} = 0$, $u \equiv v$.

Exercise 3 *(Uniqueness for a more general elliptic problem)*. Consider the linear, second-order, elliptic PDE

$$
\begin{aligned}
-\mathrm{div}(A\nabla u) + \boldsymbol{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega,\\
u &= g \quad \text{on } \partial\Omega,
\end{aligned}
\tag{2}
$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary, $A \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$, $\boldsymbol{b} \in C^1(\overline{\Omega};\mathbb{R}^n)$, and $c, f, g \in C(\overline{\Omega})$. Assume that c is non-negatives, $\mathrm{div}\boldsymbol{b} = 0$, and A is uniformly positive definite, i.e., there exists a constant $\alpha > 0$ such that $y^T A(x) y \geq \alpha |y|^2$ for all $y \in \mathbb{R}^n$, $x \in \Omega$. Prove that (2) has at most one solution $u \in C^2(\overline{\Omega})$.

Sol: Assume u_1, u_2 solve the above.
Define $v = u_1 - u_2 \in C^2(\overline{\Omega})$. v satisfies

$$
\begin{aligned}
-\mathrm{div}(A\nabla v) + \vec{b}\cdot\nabla v + cv &= 0 \quad \text{in } \Omega\\
v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

we want to show that $v \equiv 0$.
Multiplying by v and integrating gives

$$
(\ast) \quad \underbrace{-\int_\Omega v(x)\,\mathrm{div}(A(x)\nabla v(x))\,dx}_{I} + \underbrace{\int_\Omega v(x)\,\vec{b}(x)\cdot\nabla v(x)\,dx}_{II} + \int_\Omega c(x)\,v(x)^2\,dx = 0
$$

$$
v(x)\,\mathrm{div}(A(x)\nabla v(x)) = \mathrm{div}(v(x)A(x)\nabla v(x)) - \nabla v(x)^T A(x)\nabla v(x)
$$

$$
I = \int_\Omega \nabla v(x)^T A(x)\nabla v(x)\,dx \underset{\text{div thm}}{-} \int_{\partial\Omega} \cancel{v(y)A(y)\nabla v(y)\cdot n(y)}^{\,0}\,dS(y)
$$

$v|_{\partial\Omega} = 0$

$$v(x)\bar{b}(x)\cdot\bar{\nabla}v(x) = \operatorname{div}(v(x)\bar{b}(x)\,v(x)) - v(x)\operatorname{div}(v(x)\bar{b}(x))$$

$$= \operatorname{div}(v(x)^2\bar{b}(x)) - v(x)\left[b(x)\bar{\nabla}v(x) + \underbrace{\cancel{v(x)\operatorname{div}b(x)}}_{0}\right] \quad \operatorname{div}b = 0$$

$$= \operatorname{div}(v(x)^2\bar{b}(x)) - v(x)b(x)\bar{\nabla}v(x)$$

$$\to \quad v(x)b(x)\cdot\nabla v(x) = \tfrac{1}{2}\operatorname{div}(v(x)^2\bar{b}(x))$$

$$\mathrm{II} = \frac{1}{2}\int_{\partial\Omega} \underbrace{\cancel{v(y)^2\bar{b}(y)\cdot\hat{n}(y)}}_{0}\,dS(y) \qquad v|_{\partial\Omega} = 0$$

(*) becomes

$$\int_\Omega \nabla v(x)^T A(x)\nabla v(x)\,dx + \int c(x)\,v(x)^2\,dx = 0$$

see full sol!

Exercise 4. Consider the space

$$V = \{\varphi \in C^1(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega,\ \varphi \neq 0\}$$

and the functional $E : V \to \mathbb{R}$ defined by

$$E[v] = \frac{\displaystyle\int_\Omega |\nabla v(x)|^2\, dx}{\displaystyle\int_\Omega |v(x)|^2\, dx}.$$

Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E and show that

$$\begin{aligned} -\Delta u &= \lambda u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\lambda = E[u]$.

See full sol.!