PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 6 SOLUTION

In all our exercises in this sheet we will assume that Ω is an open, bounded, and connected set with smooth boundary.

Exercise 1 *(Lower bound for Dirichlet's Energy).* In this problem we will consider the Dirichlet Energy associated to the PDE

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega f(x) u(x) dx.$$

(i) Young's inequality states that for any $a, b \in \mathbb{R}$ and any Hölder conjugate numbers $p, q \in (1, \infty)$ we have that

$$|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Consequently, for any $\varepsilon > 0$ we find that by replacing a with $(p\varepsilon)^{\frac{1}{p}} a$ and b with $\frac{b}{(p\varepsilon)^{\frac{1}{p}}}$ we get that

$$|ab| \le \varepsilon |a|^p + \frac{|b|^q}{q p^{q-1} \varepsilon^{q-1}}$$

and in particular that for any $\varepsilon > 0$ choosing $p = q = 2$ yields

$$|ab| \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Show that for any $\varepsilon > 0$ and any

$$u \in V = \left\{ v \in C^1\left(\overline{\Omega}\right) \mid v = 0 \text{ on } \partial\Omega \right\}$$

we have that

$$E[u] \ge \left(\frac{1}{2} - \varepsilon C_P(\Omega)^2\right) \|u\|^2_{H^1_0(\Omega)} - \frac{1}{4\varepsilon} \|f\|^2_{L^2(\Omega)}$$

where $C_P(\Omega)$ is the Poincaré constant associated to the domain Ω.

(ii) Conclude that there exists a constant $C > 0$ such that

$$\inf_{u \in V} E[u] \ge -C.$$

Solution. (i) Using the given Young inequality we find that for any $\varepsilon > 0$

$$\left|\int_\Omega f(x)u(x)dx\right| \leq \int_\Omega \left|f(x)\right| |u(x)|\, dx$$
$$\leq \int_\Omega \left(\varepsilon |u(x)|^2 + \frac{1}{4\varepsilon}\left|f(x)\right|^2\right) dx.$$

Using the Poincaré inequality (which is allowed since $u \in V$) we find that

$$\int_\Omega f(x)u(x)dx \leq \left|\int_\Omega f(x)u(x)dx\right| \leq \varepsilon \|u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}$$
$$\leq \varepsilon C_P(\Omega)^2 \|u\|^2_{H^1_0(\Omega)} + \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}.$$

Consequently

$$E[u] = \frac{1}{2}\|u\|^2_{H^1_0(\Omega)} - \int_\Omega f(x)u(x)dx \geq \frac{1}{2}\|u\|^2_{H^1_0} - \left(\varepsilon C_P(\Omega)^2 \|u\|^2_{H^1_0(\Omega)} + \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}\right)$$
$$= \left(\frac{1}{2} - \varepsilon C_P(\Omega)^2\right)\|u\|^2_{H^1_0(\Omega)} - \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}.$$

(ii) For any $\varepsilon > 0$ such that

$$\frac{1}{2} - \varepsilon C_P(\Omega)^2 \geq 0$$

we'll find that

$$E[u] \geq -\frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)}.$$

In particular, choosing $\varepsilon = \frac{1}{2C_P(\Omega)^2}$ gives us that

$$E[u] \geq -\frac{C_P(\Omega)^2}{2}\|f\|^2_{L^2(\Omega)}.$$

□

Exercise 2 *(Uniqueness of weak solutions)*. Show that if u and v are weak solutions, in the sense defined in class, for

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

then $u = v$.

Solution. We say that u is a weak solution to the Dirichlet problem above if $u \in V = \left\{v \in C^1\left(\overline{\Omega}\right) \mid v = 0 \text{ on } \partial\Omega\right\}$ and for any $\varphi \in V$ we have that

$$\int_\Omega \nabla u(x) \cdot \nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx.$$

If u and v are both weak solutions to the equation than for any $\varphi \in V$

$$\int_\Omega \nabla u(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx$$

and

$$\int_\Omega \nabla v(x)\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx.$$

Consequently,

$$\int_\Omega \nabla\left(u(x)-v(x)\right)\cdot\nabla\varphi(x)dx = 0.$$

Since this holds for any $\varphi \in V$ we must have that $u-v=0$. Indeed, choose $\varphi = u-v$ (or $\varphi = \overline{u-v}$ if the functions are complex valued) we find that

$$\int_\Omega \left|\nabla\left(u(x)-v(x)\right)\right|^2 dx = 0,$$

which implies that $\nabla(u-v)=0$. As the domain is connected (which implies that $u-v$ is a constant) and the function $u-v$ is zero on $\partial\Omega$ we conclude that $u=v$. □

Exercise 3 *(Uniqueness for a more general elliptic problem)*. Consider the linear, second-order, elliptic PDE

$$\begin{aligned} -\mathrm{div}(A\nabla u)+\mathbf{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary, $A \in C^1(\overline{\Omega};\mathbb{R}^{n\times n})$, $\mathbf{b}\in C^1(\overline{\Omega};\mathbb{R}^n)$, and $c,f,g \in C(\overline{\Omega})$. Assume that c is non-negatives, $\mathrm{div}\mathbf{b}=0$, and A is uniformly positive definite, i.e., there exists a constant $\alpha>0$ such that $y^TA(x)y \geq \alpha|y|^2$ for all $y\in\mathbb{R}^n$, $x\in\Omega$. Prove that (2) has at most one solution $u \in C^2(\overline{\Omega})$.

Solution. Defining $u = u_1 - u_2$ we see that

$$\begin{aligned} -\mathrm{div}(A\nabla u)+\mathbf{b}\cdot\nabla u + cu &= -\mathrm{div}(A\nabla u_1)+\mathbf{b}\cdot\nabla u_1 + cu_1\\ &- \left(-\mathrm{div}(A\nabla u_2)+\mathbf{b}\cdot\nabla u_2 + cu_2\right) = f-f=0 \quad \text{in } \Omega \end{aligned}$$

and

$$u = g-g = 0, \qquad \text{on } \partial\Omega.$$

Thus, in order to show uniqueness it is enough for us to show that if u solves our equation with $f=g=0$ then u must be the zero function. Motivated by the energy method we multiply the our equation with u (or $\overline{u}$ if the function has complex values) and integrating over Ω. We find that

$$-\int_\Omega u(x)\mathrm{div}\left(A(x)\nabla u(x)\right)dx + \int_\Omega u(x)\mathbf{b}(x)\cdot\nabla u(x)dx + \int_\Omega c(x)u(x)^2dx = 0.$$

We notice that

$$-u(x)\mathrm{div}\left(A(x)\nabla u(x)\right) = -\mathrm{div}\left(u(x)A(x)\nabla u(x)\right) + \nabla u(x)\cdot A(x)\nabla u(x).$$

Similarly we have that

$$\begin{aligned} u(x)\mathbf{b}(x)\cdot\nabla u(x) &= \operatorname{div}\left(\left(u(x)\mathbf{b}(x)\right)u(x)\right) - u(x)\operatorname{div}\left(u(x)\mathbf{b}(x)\right) \\ &= \operatorname{div}\left(\mathbf{b}(x)u(x)^2\right) - u(x)\left(u(x)\operatorname{div}\mathbf{b}(x) + \mathbf{b}(x)\cdot\nabla u(x)\right) \\ &= \operatorname{div}\left(\mathbf{b}(x)u(x)^2\right) - u(x)\mathbf{b}(x)\cdot\nabla u(x), \end{aligned}$$

as $\operatorname{div}\mathbf{b} = 0$. Consequently

$$u(x)\mathbf{b}(x)\cdot\nabla u(x) = \frac{1}{2}\operatorname{div}\left(\mathbf{b}(x)u(x)^2\right).$$

Using the fact that $u|_{\partial\Omega} = 0$ we conclude that

$$\begin{aligned} -\int_\Omega u(x)\operatorname{div}\left(A(x)\nabla u(x)\right)dx &= -\int_{\partial\Omega} u(y)A(y)\nabla u(y)\cdot\hat{\mathbf{n}}(y)\,dS(y) \\ &+ \int_\Omega \nabla u(x)\cdot A(x)\nabla u(x)\,dx = \int_\Omega \nabla u(x)\cdot A(x)\nabla u(x)\,dx \end{aligned}$$

and

$$\int_\Omega u(x)\mathbf{b}(x)\cdot\nabla u(x)\,dx = \frac{1}{2}\int_{\partial\Omega} u(y)^2\mathbf{b}(y)\cdot\hat{\mathbf{n}}(y)\,dS(y) = 0.$$

Plugging these identities into our original integral yields

$$\int_\Omega \nabla u(x)\cdot A(x)\nabla u(x)\,dx + \int_\Omega c(x)u(x)^2\,dx = 0. \tag{3}$$

Using the fact that $c \geq 0$ and the uniform positive definiteness of A we see that

$$0 \leq \alpha\int_\Omega |\nabla u(x)|^2\,dx \leq \int_\Omega \nabla u(x)\cdot A(x)\nabla u(x)\,dx + \int_\Omega c(x)u(x)^2\,dx = 0,$$

which, as each term is non-negative, implies that that

$$\int_\Omega |\nabla u(x)|^2\,u\,dx = 0.$$

As this implies that $\nabla u = 0$ and since $u|_{\partial\Omega} = 0$ we conclude that $u = 0$. □

Exercise 4. Consider the space

$$V = \{\varphi \in C^1(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega,\ \varphi \neq 0\}$$

and the functional $E : V \to \mathbb{R}$ defined by

$$E[v] = \frac{\int_\Omega |\nabla v(x)|^2\,dx}{\int_\Omega |v(x)|^2\,dx}.$$

Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E and show that

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad\ \ \text{on } \partial\Omega, \end{aligned}$$

where $\lambda = E[u]$.

Solution. For any $\varphi \in V$ and any $\varepsilon > 0$ we find that $u_\varepsilon = u + \varepsilon\varphi \in C^1\left(\overline{\Omega}\right)$ and $u_\varepsilon|_{\partial\Omega} = 0$. We claim that $u_\varepsilon \neq 0$ for ε small enough. Indeed we have that

$$\left|u(x) + \varepsilon\varphi(x)\right| \geq |u(x)| - \varepsilon\left|\varphi(x)\right| \geq |u(x)| - \varepsilon\left\|\varphi\right\|_{L^\infty(\Omega)}.$$

Since $u \neq= 0$ we can find x_0 such that $|u(x_0)| > 0$. Consequently, for $\varepsilon < \frac{|u(x_0)|}{\|\varphi\|_{L^\infty(\Omega)}}$ we have that $|u_\varepsilon(x_0)| > 0$, showing that $u_\varepsilon \neq 0$.

Next we consider the function

$$g(\varepsilon) = E\left[u_\varepsilon\right]$$

We know that

$$g(0) = E[u_0] = E[u] \leq E\left[u_\varepsilon\right] = g(\varepsilon)$$

for any $\varepsilon > 0$ small enough. Consequently, if g is differentiable at $\varepsilon = 0$ we must have that $g'(0) = 0$. By the definition of E we see that

$$g(\varepsilon) = \frac{\int_\Omega \left|\nabla u(x) + \varepsilon\nabla\varphi(x)\right|^2 dx}{\int_\Omega \left|u(x) + \varepsilon\varphi(x)\right|^2 dx}$$

$$= \frac{\int_\Omega \left(|\nabla u(x)|^2 + 2\varepsilon\nabla u(x)\cdot\nabla\varphi(x) + \varepsilon^2\left|\nabla\varphi(x)\right|^2\right) dx}{\int_\Omega \left(u(x)^2 + 2\varepsilon u(x)\varphi(x) + \varepsilon^2\varphi(x)^2\right) dx},$$

which shows the required differentiability. We see that

$$g'(\varepsilon) = \frac{1}{\left(\int_\Omega \left|u(x) + \varepsilon\varphi(x)\right|^2 dx\right)^2}\left(\left(\int_\Omega \left(2\nabla u(x)\cdot\nabla\varphi(x) + 2\varepsilon\left|\nabla\varphi(x)\right|^2\right) dx\right)\int_\Omega \left|u(x) + \varepsilon\varphi(x)\right|^2 dx\right.$$

$$\left. - \left(\int_\Omega \left(2u(x)\varphi(x) + 2\varepsilon\varphi(x)^2\right) dx\right)\int_\Omega \left|\nabla u(x) + \varepsilon\nabla\varphi(x)\right|^2 dx\right)$$

and as such

$$g'(0) = \frac{2\left(\int_\Omega \nabla u(x)\cdot\nabla\varphi(x)\right)\int_\Omega u(x)^2 dx - 2\left(\int_\Omega u(x)\varphi(x)dx\right)\int_\Omega |\nabla u(x)|^2 dx}{\left(\int_\Omega u(x)^2 dx\right)^2}$$

$$= \frac{2}{\int_\Omega u(x)^2 dx}\left(\int_\Omega \left(\nabla u(x)\cdot\nabla\varphi(x) - E[u]u(x)\varphi(x)\right) dx\right).$$

Since $g'(0) = 0$ we conclude that for any $\varphi \in V$

$$\int_\Omega \left(\nabla u(x)\cdot\nabla\varphi(x) - E[u]u(x)\varphi(x)\right) dx = 0.$$

Since $u \in C^2\left(\overline{\Omega}\right)$ we have that

$$\nabla u(x)\cdot\nabla\varphi(x) = \operatorname{div}\left(\varphi(x)\nabla u(x)\right) - \varphi(x)\Delta u(x).$$

Using the divergence theorem and the fact that $\varphi|_{\partial\Omega} = 0$ we find that

$$
\begin{aligned}
0 &= \int_\Omega \left(\operatorname{div}\left(\varphi(x)\nabla u(x)\right) - \varphi(x)\Delta u(x) - E[u]\,u(x)\varphi(x)\right) dx \\
&= \int_{\partial\Omega} \varphi(y)\nabla u(y)\cdot \hat{n}(y)\, dS(y) - \int_\Omega \left(\Delta u(x) + E[u]\,u(x)\right)\varphi(x)\, dx \\
&= -\int_\Omega \left(\Delta u(x) + E[u]\,u(x)\right)\varphi(x)\, dx.
\end{aligned}
$$

This implies that

$$\int_\Omega \left(\Delta u(x) + E[u]\,u(x)\right)\varphi(x)\, dx = 0$$

for any $\varphi \in V$. Consequently, (the fundamental theorem of the calculus of variation) we must have that

$$-\Delta u(x) = E[u]\,u(x), \qquad x \in \Omega.$$

As $u \in V$ we have that $u|_\Omega = 0$ and we conclude the desired result. $\square$