PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 7

Exercise 1 *(Maximum principle for subharmonic functions)*. Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. We say that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is *subharmonic* if

$$-\Delta u \leq 0, \qquad \text{in } \Omega.$$

(i) Show that subharmonic functions satisfy the mean value formulae

$$u(x) \leq \fint_{\partial B_r(x)} u(y)\, dS(y), \qquad u(x) \leq \fint_{B_r(x)} u(y)\, dy,$$

for any $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$.

(ii) Show that subharmonic functions satisfy the strong maximum principle: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

then u is constant.

(iii) Conclude that subharmonic functions satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

(iv) Do subharmonic functions satisfy the minimum principle?

Remark: A function $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is called *superharmonic* if

$$-\Delta u \geq 0, \qquad \text{in } \Omega,$$

or equivalently if $-u$ is subharmonic. Superharmonic functions satisfy the mean value formulae

$$u(x) \geq \fint_{\partial B_r(x)} u(y)\, dS(y), \qquad u(x) \geq \fint_{B_r(x)} u(y)\, dy,$$

and the strong and weak minimum principle.

Exercise 2 *(Application of the maximum principle for subharmonic functions - comparison theorem)*. Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. Assume that for $i = 1, 2$ we have that $u_i \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases}$$

where $f_i \in C(\Omega)$ and $g_i \in C(\partial\Omega)$ for $i = 1, 2$. Assume that $f_1 \leq f_2$ and $g_1 \leq g_2$ and prove that $u_1 \leq u_2$. This is known as a *comparison principle*.

Exercise 3 *(Weak maximum principle without mean value formula)*. Let Ω be an open, bounded and connected in $\mathbb{R}^n$. Consider the equation

$$-\Delta u(x) + \boldsymbol{b}(x) \cdot \nabla u(x) = f(x), \qquad x \in \Omega$$

where $\boldsymbol{b} = \{b_i\}_{i=1}^n \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$. Our goal is to show that if $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a solution to the equation in Ω then if $f \leq 0$ (subharmonic solution) u has a weak maximum principle.

(i) Assuming $x_0 \in \Omega$ is a local maximum for u, show that if $f < 0$ on Ω then

$$-\Delta u(x_0) < 0.$$

(ii) Recall that a necessary condition for a point x_0 to be a local maximum for a C^2 function $\varphi : \Omega \to \mathbb{R}$ is that the Hessian matrix at x_0, Hess $\varphi(x_0)$, is negative semi-definite, i.e. all its eigenvalues are non-positive or equivalently, for any $\boldsymbol{y} \in \mathbb{R}^n$

$$\boldsymbol{y}^T \text{Hess}\,\varphi(x_0)\,\boldsymbol{y} \le -\alpha(x_0)\left|\boldsymbol{y}\right|^2,$$

for some $\alpha(x_0) \ge 0$. Use this to show that if $x_0 \in \Omega$ is a local maximum for u then $\Delta u(x_0) \le 0$.

(iii) Show that if $f < 0$ in Ω then u can't have a local maximum in Ω, and as such satisfies the weak maximum principle.

(iv) We want to extend the above to the case $f \le 0$. For any $\lambda, \varepsilon \in \mathbb{R}$ define

$$u_{\varepsilon,\lambda}(x) = u(x) + \varepsilon e^{\lambda x_1}.$$

(a) Show that

$$\nabla u_{\varepsilon,\lambda}(x) = \nabla u(x) + \varepsilon\lambda e^{\lambda x_1}(1,0,0,\dots,0)^T,$$

and

$$\Delta u_{\varepsilon,\lambda}(x) = \Delta u(x) + \varepsilon\lambda^2 e^{\lambda x_1}.$$

(b) Show that $u_{\varepsilon,\lambda}$ solves the equation

$$-\Delta u_{\varepsilon,\lambda}(x) + \boldsymbol{b}(x)\cdot\nabla u_{\varepsilon,\lambda}(x) = f(x) - \varepsilon\lambda\left(\lambda - b_1(x)\right)e^{\lambda x_1}.$$

(c) Show that there exists $\lambda_0 > 0$ such that for all $\varepsilon > 0$

$$f(x) - \varepsilon\lambda_0\left(\lambda_0 - b_1(x)\right)e^{\lambda_0 x_1} < 0$$

on Ω.

(d) Conclude that for any $\varepsilon > 0$

$$\max_{\overline{\Omega}}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right) = \max_{\partial\Omega}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right)$$

and consequently

$$u(x) \le \max_{\partial\Omega} u(y) + \varepsilon \max_{y\in\partial\Omega} e^{\lambda_0 y_1}$$

for all $x \in \overline{\Omega}$.

(e) Conclude the weak maximum principle for u.

Remark: This could be extended to more general elliptic PDEs.