PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 7

Exercise 1 *(Maximum principle for subharmonic functions).* Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. We say that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is *subharmonic* if

$$-\Delta u \le 0, \qquad \text{in } \Omega.$$

(i) Show that subharmonic functions satisfy the mean value formulae

$$u(x) \le \fint_{\partial B_r(x)} u(y)\, dS(y), \qquad u(x) \le \fint_{B_r(x)} u(y)\, dy,$$

for any $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$.

(ii) Show that subharmonic functions satisfy the strong maximum principle: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

then u is constant.

(iii) Conclude that subharmonic functions satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

(iv) Do subharmonic functions satisfy the minimum principle?

Remark: A function $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is called *superharmonic* if

$$-\Delta u \ge 0, \qquad \text{in } \Omega,$$

or equivalently if $-u$ is subharmonic. Superharmonic functions satisfy the mean value formulae

$$u(x) \ge \fint_{\partial B_r(x)} u(y)\, dS(y), \qquad u(x) \ge \fint_{B_r(x)} u(y)\, dy,$$

and the strong and weak minimum principle.

Sol: (i) the inequality $u(x) \le \fint_{B_r(x)} u(y)\,dy$ holds when the ineq. $u(x) \le \fint_{\partial B_r(x)} u(y)\,dy$ holds for any $r>0$ s.t. $\overline{B_r(x)} \subset \Omega$.

Indeed

$$\int_{B_r(x)} u(y)\,dy = \int_0^r \left(\int_{\partial B_\rho(x)} u(y)\,dS(y) \right) d\rho = \int_0^r \left(|\partial B_\rho(x)| \fint_{\partial B_\rho(x)} u(y)\,dS(y) \right) d\rho$$

assuming the MVT ineq. on $\partial B_\rho(x)$ $\forall \rho$

$$\geq \int_0^r n\alpha(n)\rho^{n-1} \cdot u(x)\, d\rho = |B_r(x)|\, u(x)$$

$$\rightarrow \quad u(x) \leq \fint_{B_r(x)} u(y)\, dy.$$

To show that $u(x) \leq \fint_{\partial B_r(x)} u(y)\, dS(y)$ $\forall r>0, x\in\Omega$ s.t $\overline{B_r(x)} \subset \Omega$

we define
$$\phi(\rho) = \fint_{\partial B_\rho(x)} u(y)\, dS(y)$$

we showed that since $u \in C^2(\Omega)$ and $\overline{B_r(x)} \subset \Omega$, ϕ is diff on $(0,r)$ and
$$\phi'(\rho) = \frac{\rho}{n} \fint_{B_\rho(x)} \Delta u(y)\, dy$$

As u is subharmonic $\Delta u(x) \geq 0$ $\forall x\in\Omega$ $(-\Delta u \leq 0) \Rightarrow \phi'(\rho) \geq 0$ $\forall \rho \in (0,r)$ i.e ϕ is non-decreasing on $[0,r]$.

Consequently
$$\fint_{\partial B_r(x)} u(y)\, dy = \phi(r) \underset{\forall \rho<r}{\geq} \phi(\rho) \geq \lim_{\varepsilon\to 0^+} \phi(\varepsilon)$$

$$= \lim_{\varepsilon\to 0^+} \fint_{\partial B_\varepsilon(x)} u(y)\, dy \underset{\substack{u \text{ is} \\ \text{cont.}}}{=} u(x)$$

(b) Let $M = \max_{\Omega} u$. Assume $\exists x_0 \in \Omega$ s.t

$u(x_0) = M$. Define

$$S = \{x \in \Omega \mid u(x) = M\} = u^{-1}(\{M\})$$

As Ω is connected, to show that $u \equiv M$, i.e. $S = \Omega$, it is enough to show that S is not empty, S is closed in Ω and S is open in Ω.

- $x_0 \in S$ so $S \neq \emptyset$
- u is cont. and $\{M\}$ is closed in $\mathbb{R}$ so $S = u^{-1}(\{M\})$ is closed in Ω.
- Let $x \in S$ we will find $r > 0$ s.t $B_r(x) \subset S$ showing it is open and concluding the result.

As Ω is open and $x \in \Omega$ there exist $r > 0$ s.t $B_r(x) \subset \Omega$ and consequently $B_{r/2}(x) \subset \Omega$.

$$M = u(x) \leq \fint_{B_{r/2}(x)} u(y)\,dy \leq \fint_{B_{r/2}(x)} M\,dy = M$$

$$\Rightarrow \fint_{B_{r/2}(x)} \underbrace{(M - u(y))}_{\text{cont. and non-negative}}\,dy = 0$$

which implies that $u(y) = M$ in $B_{r/2}(x)$

i.e. $B_{r/2}(x) \subset S$.

(iii) weak max principle follows from the strong principle (see lectures)

(iv) No. $u(x) = x^2$ is subharmonic on $(-1,1)$

but $\min_{[-1,1]} u(x) = u(0) \neq \min\{u(1), u(-1)\}$

Exercise 2 *(Application of the maximum principle for subharmonic functions - comparison theorem).* Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. Assume that for $i = 1,2$ we have that $u_i \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases}$$

where $f_i \in C(\Omega)$ and $g_i \in C(\partial\Omega)$ for $i = 1,2$. Assume that $f_1 \leq f_2$ and $g_1 \leq g_2$ and prove that $u_1 \leq u_2$. This is known as a *comparison principle*.

Sol: Define

$$u = u_1 - u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$$

we want to show that $u(x) \leq 0$ $\forall x \in \Omega$.

$$-\Delta u = f_1 - f_2 \leq 0 \quad \text{in } \Omega$$

$$u = g_1 - g_2 \leq 0 \quad \text{on } \partial\Omega$$

u is subharmonic $\Rightarrow$ u satisfies a max principle:

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x) = \max_{x \in \partial\Omega} (g_1(x) - g_2(x)) \leq 0$$

$$\Rightarrow \quad u(x) \leq 0 \quad \forall x \in \overline{\Omega}$$

Exercise 3 *(Weak maximum principle without mean value formula).* Let Ω be an open, bounded and connected in $\mathbb{R}^n$. Consider the equation

$$-\Delta u(x) + \boldsymbol{b}(x)\cdot\nabla u(x) = f(x), \qquad x \in \Omega$$

where $\boldsymbol{b} = \{b_i\}_{i=1}^n \in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$. Our goal is to show that if $u \in C^2(\Omega)\cap C\left(\overline{\Omega}\right)$ is a solution to the equation in Ω then if $f \le 0$ (subharmonic solution) u has a weak maximum principle.

(i) Assuming $x_0 \in \Omega$ is a local maximum for u, show that if $f < 0$ on Ω then
$$-\Delta u(x_0) < 0.$$

(ii) Recall that a necessary condition for a point x_0 to be a local maximum for a C^2 function $\varphi : \Omega \to \mathbb{R}$ is that the Hessian matrix at x_0, Hess $\varphi(x_0)$, is negative semi-definite, i.e. all its eigenvalues are non-positive or equivalently, for any $\boldsymbol{y} \in \mathbb{R}^n$
$$\boldsymbol{y}^T \text{Hess}\,\varphi(x_0)\,\boldsymbol{y} \le -\alpha(x_0)\left|\boldsymbol{y}\right|^2,$$
for some $\alpha(x_0) \ge 0$. Use this to show that if $x_0 \in \Omega$ is a local maximum for u then $\Delta u(x_0) \le 0$.

(iii) Show that if $f < 0$ in Ω then u can't have a local maximum in Ω, and as such satisfies the weak maximum principle.

(iv) We want to extend the above to the case $f \le 0$. For any $\lambda, \varepsilon \in \mathbb{R}$ define
$$u_{\varepsilon,\lambda}(x) = u(x) + \varepsilon e^{\lambda x_1}.$$

(a) Show that
$$\nabla u_{\varepsilon,\lambda}(x) = \nabla u(x) + \varepsilon\lambda e^{\lambda x_1}(1,0,0,\dots,0)^T,$$
and
$$\Delta u_{\varepsilon,\lambda}(x) = \Delta u(x) + \varepsilon\lambda^2 e^{\lambda x_1}.$$

(b) Show that $u_{\varepsilon,\lambda}$ solves the equation
$$-\Delta u_{\varepsilon,\lambda}(x) + \boldsymbol{b}(x)\nabla u_{\varepsilon,\lambda}(x) = f(x) - \varepsilon\lambda(\lambda - b_1(x))\,e^{\lambda x_1}.$$

(c) Show that there exists $\lambda_0 > 0$ such that for all $\varepsilon > 0$
$$f(x) - \varepsilon\lambda_0(\lambda_0 - b_1(x))\,e^{\lambda_0 x_1} < 0$$
on Ω.

(d) Conclude that for any $\varepsilon > 0$
$$\max_{\overline{\Omega}}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right) = \max_{\partial\Omega}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right)$$
and consequently
$$u(x) \le \max_{\partial\Omega} u(y) + \varepsilon \max_{y\in\partial\Omega} e^{\lambda_0 y_1}$$
for all $x \in \overline{\Omega}$.

(e) Conclude the weak maximum principle for u.

Remark: This could be extended to more general elliptic PDEs.

Sol:

(i) if $x_0 \in \Omega$ is a local max (Ω is open) $\Rightarrow$ $\nabla u(x_0) = \bar{0}$

Thus

$$-\Delta u(x_0) + \underbrace{\cancel{\bar{b}(x_0)\cdot\nabla u(x_0)}}_{0} = f(x_0) < 0$$

and we get

$$-\Delta u(x_0) < 0$$

(ii) $\Delta u(x_0) = \sum_{i=1}^{n} u_{x_i x_i}(x_0) = \mathrm{Tr}\,(\mathrm{Hess}\; u)(x_0)$

$= \sum$ eigenvalues of $\mathrm{Hess}\,u(x_0)$ ≤ 0

since all eigenvalues are non-positive as x_0 is a local max.

(iii) Had there been $x_0 \in \Omega$ that is a local max when $f<0$ then by (i)

$$-\Delta u(x_0) < 0$$

and by (ii) $\Delta u(x_0) \leq 0$

contradiction.

Consequently u can't attain its global max ~~of~~ $\cancel{\overline{\Omega}}$ in Ω i.e

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

<u>Rest</u>: see full solution.