## PARTIAL DIFFERENTIAL EQUATIONS III & V PROBLEM CLASS 7

**Exercise 1** (*Maximum principle for subharmonic functions*). Let  $\Omega$  be an open, bounded and connected set in  $\mathbb{R}^n$ . We say that  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is *subharmonic* if

$$-\Delta u \leq 0$$
, in  $\Omega$ .

(i) Show that subharmonic functions satisfy the mean value formulae

$$u(x) \leq \int_{\partial B_r(x)} u(y) dS(y), \qquad u(x) \leq \int_{B_r(x)} u(y) dy,$$

for any  $x \in \Omega$  and r > 0 such that  $\overline{B_r(x)} \subset \Omega$ .

(ii) Show that subharmonic functions satisfy the strong maximum principle: If there exists  $x_0 \in \Omega$  such that

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

then *u* is constant.

(iii) Conclude that subharmonic functions satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u(x) = \max_{\partial \Omega} u(x).$$

(iv) Do subharmonic functions satisfy the minimum principle?

**Remark:** A function  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is called *superharmonic* if

$$-\Delta u \ge 0, \quad \text{in } \Omega$$

or equivalently if -u is subharmonic. Superharmonic functions satisfy the mean value formulae

$$u(x) \ge \int_{\partial B_r(x)} u(y) dS(y), \qquad u(x) \ge \int_{B_r(x)} u(y) dy,$$

and the strong and weak minimum principle.

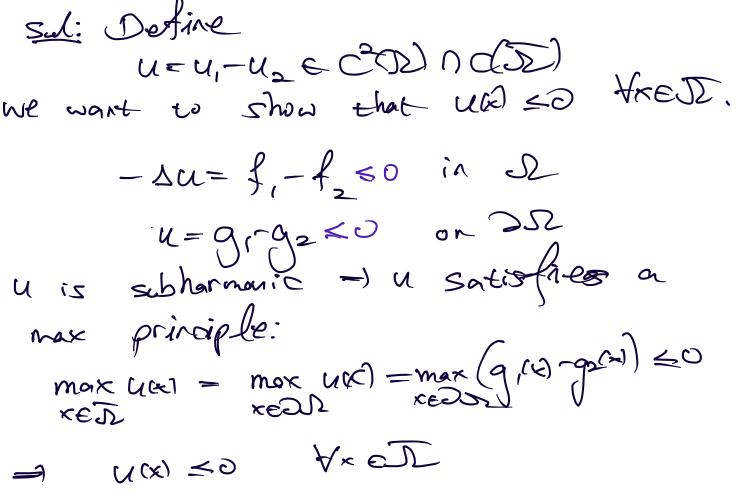
PROBLEM CLASS 7 u(x)= ll. Define  $S = dx \in D | u(x) = l(f) = u(f))$ As I is connected, to show that le=M , i.e. S=J, it is enough to show that s is not empty, S is absel in I and s is open in J2. . RES SO SEE . U is cont. and 2023 is closed in R so  $S = u^{r}(\{M\})$  is closed in N. · Let xeS we will find 1-22 s.t Braics showing it is open and concluding the result. As I is open and xEI there exist noo st Brach and consequency Brach CD  $M = U(x) \leq \int U(y) dy = M$   $B_{n_0}(x) = B_{n_0}(x)$ =) f(u-ukg))dy = 0 By(x) care and non-negative which implies ungs=Il in Bright ie Brainces.

(m) we	are max principle fellows
from	the strong principle (see lectures)
(iv) No.	usa=x² is subharmonic on (-1,1)
but	min $u(x) = u(x) \neq min \{u(x), u(-x)\}$ $\Sigma - i, i$

**Exercise 2** (Application of the maximum principle for subharmonic functions - comparison theorem). Let  $\Omega$  be an open, bounded and connected set in  $\mathbb{R}^n$ . Assume that for i = 1, 2 we have that  $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$  satisfy

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial \Omega, \end{cases}$$

where  $f_i \in C(\Omega)$  and  $g_i \in C(\partial\Omega)$  for i = 1, 2. Assume that  $f_1 \leq f_2$  and  $g_1 \leq g_2$  and prove that  $u_1 \leq u_2$ . This is known as a *comparison principle*.



**Exercise 3** (*Weak maximum principle without mean value formula*). Let  $\Omega$  be an open, bounded and connected in  $\mathbb{R}^n$ . Consider the equation

$$-\Delta u(x) + \boldsymbol{b}(x) \cdot \nabla u(x) = f(x), \qquad x \in \Omega$$

where  $\boldsymbol{b} = \{b_i\}_{i=1}^n \in C^1(\overline{\Omega}; \mathbb{R}^n)$ . Our goal is to show that if  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is a solution to the equation in  $\Omega$  then if  $f \leq 0$  (subharmonic solution) u has a weak maximum principle.

(i) Assuming  $x_0 \in \Omega$  is a local maximum for *u*, show that if f < 0 on  $\Omega$  then

$$-\Delta u(x_0) < 0.$$

(ii) Recall that a necessary condition for a point  $x_0$  to be a local maximum for a  $C^2$  function  $\varphi : \Omega \to \mathbb{R}$  is that the Hessian matrix at  $x_0$ , Hess  $\varphi(x_0)$ , is negative semi-definite, i.e. all its eigenvalues are non-positive or equivalently, for any  $y \in \mathbb{R}^n$ 

$$\mathbf{y}^T$$
 Hess  $\varphi(x_0) \mathbf{y} \leq -\alpha(x_0) |\mathbf{y}|^2$ ,

for some  $\alpha(x_0) \ge 0$ . Use this to show that if  $x_0 \in \Omega$  is a local maximum for *u* then  $\Delta u(x_0) \le 0$ . (iii) Show that if f < 0 in  $\Omega$  then *u* can't have a local maximum in  $\Omega$ , and as such satisfies the

- weak maximum principle.
- (iv) We want to extend the above to the case  $f \le 0$ . For any  $\lambda, \varepsilon \in \mathbb{R}$  define

$$u_{\varepsilon,\lambda}(x) = u(x) + \varepsilon e^{\lambda x_1}$$

(a) Show that

$$\nabla u_{\varepsilon,\lambda}(x) = \nabla u(x) + \varepsilon \lambda e^{\lambda x_1} (1, 0, 0, \dots, 0)^T,$$

and

$$\Delta u_{\varepsilon,\lambda}(x) = \Delta u(x) + \varepsilon \lambda^2 e^{\lambda x_1}.$$

(b) Show that  $u_{\varepsilon,\lambda}$  solves the equation

$$-\Delta u_{\varepsilon,\lambda}(x) + \boldsymbol{b}(x) \, \boldsymbol{\nabla} u_{\varepsilon,\lambda}(x) = f(x) - \varepsilon \lambda \, (\lambda - b_1(x)) \, e^{\lambda x_1}.$$

(c) Show that there exists  $\lambda_0 > 0$  such that for all  $\varepsilon > 0$ 

$$f(x) - \varepsilon \lambda_0 \left(\lambda_0 - b_1(x)\right) e^{\lambda_0 x_1} < 0$$

on Ω.

(d) Conclude that for any  $\varepsilon > 0$ 

$$\max_{\overline{\Omega}} \left( u(x) + \varepsilon e^{\lambda_0 x_1} \right) = \max_{\partial \Omega} \left( u(x) + \varepsilon e^{\lambda_0 x_1} \right)$$

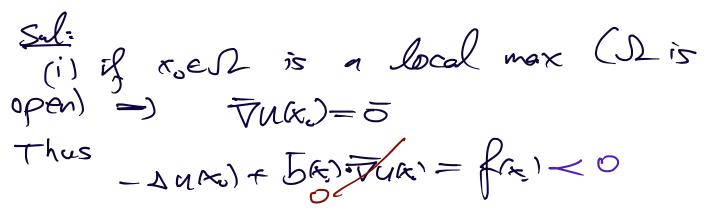
and consequently

$$u(x) \le \max_{\partial \Omega} u(y) + \varepsilon \max_{y \in \partial \Omega} e^{\lambda_0 y_1}$$

for all  $x \in \overline{\Omega}$ .

(e) Conclude the weak maximum principle for *u*.

Remark: This could be extended to more general elliptic PDEs.



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