# PARTIAL DIFFERENTIAL EQUATIONS III & V
# PROBLEM CLASS 7 SOLUTION

**Exercise 1** *(Maximum principle for subharmonic functions).* Let $\Omega$ be an open, bounded and connected set in $\mathbb{R}^n$. We say that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is *subharmonic* if

$$-\Delta u \leq 0, \qquad \text{in } \Omega.$$

(i) Show that subharmonic functions satisfy the mean value formulae

$$u(x) \leq \fint_{\partial B_r(x)} u(y)\,dS(y), \qquad u(x) \leq \fint_{B_r(x)} u(y)\,dy,$$

for any $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$.

(ii) Show that subharmonic functions satisfy the strong maximum principle: If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x)$$

then $u$ is constant.

(iii) Conclude that subharmonic functions satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

(iv) Do subharmonic functions satisfy the minimum principle?

**Remark:** A function $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is called *superharmonic* if

$$-\Delta u \geq 0, \qquad \text{in } \Omega,$$

or equivalently if $-u$ is subharmonic. Superharmonic functions satisfy the mean value formulae

$$u(x) \geq \fint_{\partial B_r(x)} u(y)\,dS(y), \qquad u(x) \geq \fint_{B_r(x)} u(y)\,dy,$$

and the strong and weak minimum principle.

*Solution.* (i) Much like in class, we will show that the second mean value theorem follows from the first. Indeed, given $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$ we see that $\overline{B_\rho(x)} \subset \Omega$ for all $\rho \in (0, r]$ and

$$\int_{B_r(x)} u(y)\,dy = \int_0^r \int_{\partial B_\rho(x)} u(y)\,dS(y)\,d\rho \underset{\text{under our assumption}}{\geq}$$

$$\int_0^r \left|\partial B_\rho(x)\right| u(x)\,d\rho = n\alpha(n)u(x)\int_0^r \rho^{n-1}\,d\rho = |B_r(x)|\,u(x),$$

which implies that

$$u(x) \leq \fint_{B_r(x)} u(y)\,dy.$$

We turn our attention to the first mean value theorem and consider the function

$$\phi(\rho) = \fint_{\partial B_\rho(x)} u(y)\,dS(y).$$

Just like in class, since $u \in C^2(\Omega)$ we see that for any $x \in \Omega$, $\phi(\rho)$ is differentiable on an open interval $(0, r)$ and

$$\phi'(\rho) = \frac{\rho}{n}\fint_{B_\rho(x)} \Delta u(y)\,dy.$$

As $u$ is subharmonic we conclude that $\phi'(\rho) \geq 0$ and consequently that $\phi$ is increasing. We conclude that for any $r > 0$ such that $\overline{B_r(x)} \subset \Omega$ we have that

$$\fint_{\partial B_r(x)} u(y)\,dS(y) = \phi(r) \geq \lim_{\varepsilon \to 0^+} \phi(\varepsilon) = u(x).$$

Here we have used the fact that if $f$ is continuous at $x_0$ then

$$\fint_{\partial B_\varepsilon(x_0)} f(y)\,dy \underset{\varepsilon \to 0^+}{\longrightarrow} f(x_0).$$

(ii) The proof is almost identical to the strong maximum principle from class. Assume that $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u(x) = M.$$

Then, since $\Omega$ is open, we can find $r(x_0) > 0$ such that $\overline{B_{r(x_0)}}(x_0) \subset \Omega$. Since $u$ is subharmonic the mean value formula tells us that

$$M = u(x_0) \leq \fint_{B_{r(x_0)}(x_0)} u(y)\,dy \leq \fint_{B_{r(x_0)}(x_0)} M\,dy = M.$$

This implies that $\fint_{B_{r(x_0)}(x_0)} u(y)\,dy = M$ or that

$$\fint_{B_{r(x_0)}(x_0)} \big(M - u(y)\big)\,dy = 0.$$

Since $M - u(y) \geq 0$ for all $y \in \Omega$ and it is a continuous function on $B_r(x_0)$ we must have that $u(y) = M$ for all $y \in B_{r(x_0)}(x_0)$. We conclude that if $x_0 \in u^{-1}\{M\} \cap \Omega$ then there exists $r(x_0) > 0$ such that $B_{r(x_0)}(x_0) \subset u^{-1}(M) \cap \Omega$. i.e. the set $u^{-1}\{M\} \cap \Omega$ is open in $\Omega$. It is also closed in $\Omega$ as the preimage of a closed set by a continuous function. Since $\Omega$ is connected and $u^{-1}\{M\} \cap \Omega$ is not empty by assumption, we conclude that it is $\Omega$, i.e. $u$ is constant on $\Omega$. Due to continuity of $u$ we conclude that it is also constant on $\overline{\Omega}$ which shows the desired result.

(iii) This follows directly as in class - Since $u \in C\left(\overline{\Omega}\right)$ it must have a maximum on $\overline{\Omega}$. Denote it by $M$. If it is attained at an internal point then $u$ must be constant and as such

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

Otherwise, it is attained on $\partial\Omega$ and the above still holds.

(iv) A subharmonic function does not satisfy the minimum principle necessarily. Indeed, consider the one dimensional function $u(x) = x^2$ on $(-1, 1)$. It is subharmonic but its minimum is an internal point.

$\square$

**Exercise 2** *(Application of the maximum principle for subharmonic functions - comparison theorem).* Let $\Omega$ be an open, bounded and connected set in $\mathbb{R}^n$. Assume that for $i = 1,2$ we have that $u_i \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases}$$

where $f_i \in C(\Omega)$ and $g_i \in C(\partial\Omega)$ for $i = 1,2$. Assume that $f_1 \leq f_2$ and $g_1 \leq g_2$ and prove that $u_1 \leq u_2$. This is known as a *comparison principle.*

*Solution.* We define $u = u_1 - u_2 \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and notice that in $\Omega$

$$-\Delta u = f_1 - f_2 \leq 0,$$

i.e. $u$ is subharmonic. Using the weak maximum principle we find that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u = \max_{\partial\Omega} \left(g_1 - g_2\right) \leq 0,$$

which implies that for any $x \in \overline{\Omega}$

$$u_1(x) - u_2(x) = u(x) \leq 0,$$

showing the desired result. $\square$

**Exercise 3** *(Weak maximum principle without mean value formula).* Let $\Omega$ be an open, bounded and connected in $\mathbb{R}^n$. Consider the equation

$$-\Delta u(x) + \boldsymbol{b}(x) \cdot \nabla u(x) = f(x), \qquad x \in \Omega$$

where $\boldsymbol{b} = \{b_i\}_{i=1}^n \in C^1\left(\overline{\Omega}; \mathbb{R}^n\right)$. Our goal is to show that if $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a solution to the equation in $\Omega$ then if $f \leq 0$ (subharmonic solution) $u$ has a weak maximum principle.

(i) Assuming $x_0 \in \Omega$ is a local maximum for $u$, show that if $f < 0$ on $\Omega$ then

$$-\Delta u(x_0) < 0.$$

(ii) Recall that a necessary condition for a point $x_0$ to be a local maximum for a $C^2$ function $\varphi : \Omega \to \mathbb{R}$ is that the Hessian matrix at $x_0$, Hess $\varphi(x_0)$, is negative semi-definite, i.e. all its eigenvalues are non-positive or equivalently, for any $\boldsymbol{y} \in \mathbb{R}^n$

$$\boldsymbol{y}^T \text{Hess}\, \varphi(x_0)\, \boldsymbol{y} \leq -\alpha(x_0) \left|\boldsymbol{y}\right|^2,$$

for some $\alpha(x_0) \geq 0$. Use this to show that if $x_0 \in \Omega$ is a local maximum for $u$ then $\Delta u(x_0) \leq 0$.

(iii) Show that if $f < 0$ in $\Omega$ then $u$ can't have a local maximum in $\Omega$, and as such satisfies the weak maximum principle.

(iv) We want to extend the above to the case $f \leq 0$. For any $\lambda, \varepsilon \in \mathbb{R}$ define

$$u_{\varepsilon,\lambda}(x) = u(x) + \varepsilon e^{\lambda x_1}.$$

(a) Show that

$$\nabla u_{\varepsilon,\lambda}(x) = \nabla u(x) + \varepsilon\lambda e^{\lambda x_1}(1, 0, 0, \ldots, 0)^T,$$

and

$$\Delta u_{\varepsilon,\lambda}(x) = \Delta u(x) + \varepsilon\lambda^2 e^{\lambda x_1}.$$

(b) Show that $u_{\varepsilon,\lambda}$ solves the equation

$$-\Delta u_{\varepsilon,\lambda}(x) + \boldsymbol{b}(x) \cdot \nabla u_{\varepsilon,\lambda}(x) = f(x) - \varepsilon\lambda\left(\lambda - b_1(x)\right) e^{\lambda x_1}.$$

(c) Show that there exists $\lambda_0 > 0$ such that for all $\varepsilon > 0$

$$f(x) - \varepsilon\lambda_0 (\lambda_0 - b_1(x))\, e^{\lambda_0 x_1} < 0$$

on $\Omega$.

(d) Conclude that for any $\varepsilon > 0$

$$\max_{\overline{\Omega}} \left( u(x) + \varepsilon e^{\lambda_0 x_1} \right) = \max_{\partial\Omega} \left( u(x) + \varepsilon e^{\lambda_0 x_1} \right)$$

and consequently

$$u(x) \leq \max_{\partial\Omega} u(y) + \varepsilon \max_{y\in\partial\Omega} e^{\lambda_0 y_1}$$

for all $x \in \overline{\Omega}$.

(e) Conclude the weak maximum principle for $u$.

**Remark:** This could be extended to more general elliptic PDEs.

*Solution.* (i) We know that if $x_0 \in \Omega$ is a local extremum then $\nabla u(x_0) = 0$. Consequently,

$$-\Delta u(x_0) = f(x_0) < 0.$$

(ii) We recall that

$$\Delta u(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x) = \operatorname{tr}(\operatorname{Hess} u(x)).$$

As the trace of a symmetric matrix is the sum of its eigenvalues, we know that if $x_0$ is a local maximum then these eigenvalues are non-positive and consequently

$$\Delta u(x_0) \leq 0.$$

(iii) This follows immediately from the last two parts. Indeed, if $x_0$ was local maximum then $-\Delta u(x_0) < 0$ and $\Delta u(x_0) \leq 0$ which is impossible. Consequently, $u$ can't attain any global maximum in an internal point and we get the weak maximum principle.

(iv) (a) We have that

$$\partial_{x_i} u_{\varepsilon,\lambda}(x) = \partial_{x_i} u(x) + \begin{cases} \varepsilon\lambda e^{\lambda x_1}, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

which shows the first statement. Similarly

$$\partial_{x_i x_i} u_{\varepsilon,\lambda}(x) = \partial_{x_i x_i} u(x) + \begin{cases} \varepsilon\lambda^2 e^{\lambda x_1}, & i = 1, \\ 0, & i \neq 1, \end{cases}$$

and as such

$$\Delta u_{\varepsilon,\lambda}(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x) = \Delta u(x) + \varepsilon\lambda^2 e^{\lambda x}.$$

(b) We have that

$$-\Delta u_{\varepsilon,\lambda}(x) + \mathbf{b}(x)\cdot\nabla u_{\varepsilon,\lambda}(x) = -\Delta u(x) - \varepsilon\lambda^2 e^{\lambda x_1} + \mathbf{b}(x)\cdot\nabla u(x)$$
$$+\varepsilon\lambda e^{\lambda x_1}\mathbf{b}(x)\cdot(1,0,0,\dots,0)^T = f(x) - \varepsilon\lambda^2 e^{\lambda x_1} + \varepsilon\lambda b_1(x) e^{\lambda x_1},$$

from which we get the desired equality.

(c) Choosing any $\lambda > \|b_1\|_{L^\infty\left(\overline{\Omega}\right)}$ will give us the desired result. For instance we can choose $\lambda_0 = \|b_1\|_{L^\infty\left(\overline{\Omega}\right)} + 1$.

(d) Since

$$-\Delta u_{\varepsilon,\lambda_0}(x) + \mathbf{b}(x)\cdot\nabla u_{\varepsilon,\lambda_0}(x) = f(x) - \varepsilon\lambda_0\left(\lambda_0 - b_1(x)\right)e^{\lambda_0 x_1}$$
$$\leq 0 - \varepsilon\lambda_0\left(\lambda_0 - b_1(x)\right)e^{\lambda_0 x_1} < 0$$

on $\Omega$ we conclude from (iii) that

$$\max_{\overline{\Omega}}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right) = \max_{\partial\Omega}\left(u(x) + \varepsilon e^{\lambda_0 x_1}\right).$$

Consequently, for any $\varepsilon > 0$ and any $x \in \overline{\Omega}$

$$u(x) \leq u(x) + \varepsilon e^{\lambda_0 x_1} \leq \max_{\overline{\Omega}}\left(u(y) + \varepsilon e^{\lambda_0 y_1}\right)$$
$$= \max_{\partial\Omega}\left(u(y) + \varepsilon e^{\lambda_0 y_1}\right) \leq \max_{\partial\Omega} u(y) + \varepsilon \max_{\partial\Omega} e^{\lambda_0 y_1}.$$

(e) As the above holds for any $\varepsilon > 0$ we can take it to zero to conclude that for any $x \in \overline{\Omega}$

$$u(x) \leq \max_{\partial\Omega} u(y).$$

Consequently

$$\max_{\partial\Omega} u(x) \leq \max_{\overline{\Omega}} u(x) \leq \max_{\partial\Omega} u(x),$$

which gives us the desired weak maximum principle

$$\max_{\overline{\Omega}} u(x) = \max_{\partial\Omega} u(x).$$

$\square$