PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 8

Exercise 1 *(Properties of the fundamental solution to the Heat equation)*. Let

$$\Phi(x,t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}},$$

where $k > 0$ is given, be the fundamental solution to the Heat equation.

(i) Show that for any $p \geq 1$

$$\|\Phi(x,t)\|_{L^p(\mathbb{R}^n)} = \frac{(4\pi k t)^{\frac{n(1-p)}{2p}}}{p^{\frac{n}{2p}}}$$

You may use the fact that $\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2}} dy = (2\pi)^{\frac{n}{2}}$.

In particular,

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = \|\Phi(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$$

for all $t > 0$.

(ii) Young's convolution inequality states that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $p, q \in [1,\infty]$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

for some $r \in [1,\infty]$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\left\|f * g\right\|_{L^r(\mathbb{R}^n)} \leq \left\|f\right\|_{L^p(\mathbb{R}^n)} \left\|g\right\|_{L^q(\mathbb{R}^n)}.$$

Use this to show that for any $g \in C_c(\mathbb{R}^n)$ we have that for any $r \geq 1$ and any $1 \leq p \leq r$

$$\left\|\Phi(\cdot,t) * g\right\|_{L^r} \leq \frac{C_{p,n,k} \left\|g\right\|_{L^{\frac{rp}{rp+p-r}}(\mathbb{R}^n)}}{t^{\frac{n(p-1)}{2p}}}$$

where $C_{p,n,k}$ is an explicit constant that depends only on p, n, and k.

Exercise 2 *(Bonus – additional bounds on $u = \Phi * g$)*. Consider the solution to the heat equation

$$\begin{aligned} &u_t(x,t) - k\Delta u(x,t) = 0, && x \in \mathbb{R}^n,\ t > 0,\\ &u(x,0) = g(x), && x \in \mathbb{R}^n, \end{aligned}$$

where $k > 0$ and $g \in L^1(\mathbb{R}^n)$, given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy$$

with

$$\Phi(x,t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}}.$$

(i) Show that if $g \in L^\infty(\mathbb{R}^n)$ then so is u. Moreover

$$\|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))} \leq \left\|g\right\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) If $g \in L^2(\mathbb{R}^n)$ one can show that $u(\cdot,t) \in L^2(\mathbb{R}^n)$ for all $t > 0$ (follows from the previous exercise!). Show that for any $t > 0$

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \|g\|_{L^2(\mathbb{R}^n)}.$$

Hint: You may use the following:

$$\Phi \ge 0, \qquad \text{and} \qquad \int_{\mathbb{R}^n} \Phi(z,t)\,dz = 1, \ \ \forall t > 0.$$

$$\widehat{\Phi}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-kt|\xi|^2}.$$

Exercise 3 *(The energy method: Uniqueness for the heat equation in a time dependent domain).* Let $k > 0$, $T > 0$ be given. Let $a, b : [0,T] \to \mathbb{R}$ be smooth functions such that $a(t) < b(t)$ for all $t \in [0,T]$. Let $U \subset \mathbb{R} \times (0,T]$ be the non-cylindrical domain

$$U = \{(x,t) \in \mathbb{R} \times (0,T] \mid a(t) < x < b(t)\}.$$

Consider the heat equation

$$\begin{cases} u_t - ku_{xx} = f(x,t) & (x,t) \in U, \\ u(a(t),t) = g_1(t) & t \in [0,T], \\ u(b(t),t) = g_2(t) & t \in [0,T], \\ u(x,0) = u_0(x) & x \in (a(0),b(0)). \end{cases}$$

Use the energy method to prove that the equation has at most one smooth solution.

Exercise 4 *(Grönwall's inequality).* Another important inequality in the study of PDEs, in particular in the study of long time behaviour of solutions, is the so-called Grönwall's inequality which state that if $y : [0,T] \to \mathbb{R}$ is continuous and differentiable on $(0,T)$, and if there exists $\lambda \in \mathbb{R}$ such that

$$y'(t) \le \lambda y(t)$$

then we have that

$$y(t) \le y(0) e^{\lambda t}.$$

Prove Grönwall's inequality.
Remark: The above inequality can be generalised to show that if

$$y'(t) \le \lambda(t) y(t)$$

then $y(t) \le y(0) e^{\int_0^t \lambda(s)ds}$. There are additional important Grönwall inequalities which we won't mention at this point.

Exercise 5. Let $T > 0$ be given and define

$$\Omega_T = (a,b) \times (0,T]$$

for a given $-\infty < a < b < \infty$.

(i) Show that there exists at most one solution $u \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ to the problem

$$\begin{cases} u_t - u_{xx} = 1 & (x,t) \in \Omega_T, \\ u = 0 & (x,t) \in \Gamma_T, \end{cases} \tag{1}$$

where $\Gamma_T = (a,b) \times \{0\} \cup \{a\} \times [0,T] \cup \{b\} \times [0,T]$.

(ii) Assume that u is a $C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ solution to (1). Show that for any $(x,t) \in \Omega_T$

$$0 \le u(x,t) \le t.$$