PARTIAL DIFFERENTIAL EQUATIONS III & V
PROBLEM CLASS 8

Exercise 1 *(Properties of the fundamental solution to the Heat equation).* Let

$$\Phi(x,t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}},$$

where $k > 0$ is given, be the fundamental solution to the Heat equation.

(i) Show that for any $p \geq 1$

$$\|\Phi(x,t)\|_{L^p(\mathbb{R}^n)} = \frac{(4\pi k t)^{\frac{n(1-p)}{2p}}}{p^{\frac{n}{2p}}}$$

You may use the fact that $\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2}} dy = (2\pi)^{\frac{n}{2}}$.

In particular,

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = \|\Phi(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$$

for all $t > 0$.

(ii) Young's convolution inequality states that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $p, q \in [1,\infty]$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

for some $r \in [1,\infty]$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Use this to show that for any $g \in C_c(\mathbb{R}^n)$ we have that for any $r \geq 1$ and any $1 \leq p \leq r$

$$\|\Phi(\cdot,t) * g\|_{L^r} \leq \frac{C_{p,n,k} \|g\|_{L^{\frac{rp}{rp+p-r}}(\mathbb{R}^n)}}{t^{\frac{n(p-1)}{2p}}}$$

where $C_{p,n,k}$ is an explicit constant that depends only on p, n, and k.

See full sol.

Exercise 2 *(Bonus – additional bounds on $u = \Phi * g$)*. Consider the solution to the heat equation

$$\begin{aligned} &u_t(x,t) - k\Delta u(x,t) = 0, && x \in \mathbb{R}^n,\ t > 0,\\ &u(x,0) = g(x), && x \in \mathbb{R}^n, \end{aligned}$$

$$\cancel{u_t(x,t) - k\Delta u(x,t) = 0, \quad x \in \mathbb{R}^n,\ t > 0,}$$
$$\cancel{u(x,0) = g(x), \quad x \in \mathbb{R}^n,}$$

where $k > 0$ and $g \in L^1(\mathbb{R}^n)$, given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy$$

with

$$\Phi(x,t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}}.$$

(i) Show that if $g \in L^\infty(\mathbb{R}^n)$ then so is u. Moreover

$$\|u\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \le \|g\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) If $g \in L^2(\mathbb{R}^n)$ one can show that $u(\cdot,t) \in L^2(\mathbb{R}^n)$ for all $t > 0$ (follows from the previous exercise!). Show that for any $t > 0$

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \|g\|_{L^2(\mathbb{R}^n)}.$$

Hint: You may use the following:

$$\Phi \ge 0, \qquad \text{and} \qquad \int_{\mathbb{R}^n} \Phi(z,t)\,dz = 1,\ \ \forall t > 0.$$

$$\widehat{\Phi}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-kt|\xi|^2}.$$

Solution:

(i)
$$|u(x,t)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy \right|$$

$$\underset{\Phi \ge 0}{\le} \int_{\mathbb{R}^n} \Phi(x-y,t)\, |g(y)|\, dy \underset{\Phi \ge 0}{\le} \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi(x-y,t)\, dy$$

$$\underset{\substack{z = x-y \\ dz = |(-1)^n|\, dy}}{=} \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi(z,t)\, dz = \|g\|_{L^\infty(\mathbb{R}^n)}$$

$$\Rightarrow \quad \|u\|_{L^\infty(\mathbb{R}^n \times (0,\infty))} \le \|g\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) By Plancherel's identity

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$$

$$\hat u(\xi,t) = \widehat{\Phi(\cdot,t) * g(\cdot)}(\xi) \underset{\substack{\uparrow \\ \text{convolution} \\ \text{then}}}{=} (2\pi)^{n/2}\, \hat\Phi(\xi,t)\, \hat g(\xi)$$

$$= \hat g(\xi)\, e^{-kt|\xi|^2}$$

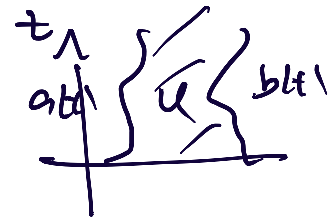
So

$$\int_{\mathbb{R}^n} |\hat u(\xi,t)|^2 \, d\xi = \int_{\mathbb{R}^n} |\hat g(\xi)|^2 e^{-2kt|\xi|^2} \, d\xi$$

$$\le \underbrace{\int_{\mathbb{R}^n} |\hat g(\xi)|^2 \, d\xi}_{\|\hat g\|^2_{L^2(\mathbb{R}^n)}} \underset{\text{Plancherel}}{=} \|g\|^2_{L^2(\mathbb{R}^n)}$$

Exercise 3 *(The energy method: Uniqueness for the heat equation in a time dependent domain).* Let $k > 0$, $T > 0$ be given. Let $a, b : [0, T] \to \mathbb{R}$ be smooth functions such that $a(t) < b(t)$ for all $t \in [0, T]$. Let $U \subset \mathbb{R} \times (0, T]$ be the non-cylindrical domain

$$U = \{(x,t) \in \mathbb{R} \times (0,T] \mid a(t) < x < b(t)\}.$$



Consider the heat equation

$$\begin{cases} u_t - ku_{xx} = f(x,t) & (x,t) \in U, \\ u(a(t),t) = g_1(t) & t \in [0,T], \\ u(b(t),t) = g_2(t) & t \in [0,T], \\ u(x,0) = u_0(x) & x \in (a(0), b(0)). \end{cases}$$

Use the energy method to prove that the equation has at most one smooth solution.

Solution: Assume there exist two smooth sol. u_1, u_2. Define $w = u_1 - u_2$. w satisfies

$$\begin{aligned} & w_t - k w_{xx} = 0 && U \\ & w(a(t),t) = 0 && t \in [0,T] \\ & w(b(t),t) = 0 && t \in [0,T] \\ & w(x,0) = 0 && x \in (a(0), b(0)) \end{aligned}$$

multiply the first eq by $w(x,t)$ and integrating over $[a(t), b(t)]$.

$$\int_{a(t)}^{b(t)} w_t(x,t) w(x,t)\,dx = k \int_{a(t)}^{b(t)} w_{xx}(x,t) w(x,t)\,dx$$

$$\int_{a(t)}^{b(t)} w_{xx}(x,t) w(x,t)\,dx = \cancel{w_x(x,t) w(x,t)\Big|_{a(t)}^{b(t)}} - \int_{a(t)}^{b(t)} w_x(x,t)^2\,dx$$

0

$w(a(t),t) = w(b(t),t) = 0$

$$= -\int_{a(t)}^{b(t)} w_x(x,t)^2\,dx$$

$$\int_{a(t)}^{b(t)} w_t(x,t)\, w(x,t)\, dx = \frac{1}{2}\int_{a(t)}^{b(t)} \frac{\partial}{\partial t} w^2(x,t)\, dx$$

Note:

$$\frac{d}{dt}\left(\frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\, dx\right) = \frac{1}{2}\int_{a(t)}^{b(t)} \frac{\partial}{\partial t} w^2(x,t)\, dx$$

$$+ \frac{1}{2} w^2\underbrace{\cancel{(b(t),t)}}_{0} \cdot b'(t) - \frac{1}{2} w^2\underbrace{\cancel{(a(t),t)}}_{0} \cdot a'(t)$$

$$= \frac{1}{2}\int_{a(t)}^{b(t)} \frac{\partial}{\partial t} w^2(x,t)\, dx$$

Defining $E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\, dx$

we find that

$$\frac{d}{dt} E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} \frac{\partial}{\partial t} w^2(x,t)\, dx = -k\int_{a(t)}^{b(t)} w_x^2(x,t)\, dx \leq 0$$

$\Rightarrow$ E is non-increasing

$\forall t>0$ $\quad \underbrace{E(t)}_{\frac{1}{2}\int w^2(x,t)\, dx} \leq E(0) = \frac{1}{2}\int_{a(0)}^{b(0)} w^2(x,0)\, dx = 0$

so $\quad w(x,t) = 0 \quad \forall x,t \in \mathcal{U}$

Exercise 4 *(Grönwall's inequality).* Another important inequality in the study of PDEs, in particular in the study of long time behaviour of solutions, is the so-called Grönwall's inequality which state that if $y : [0, T] \to \mathbb{R}$ is continuous and differentiable on $(0, T)$, and if there exists $\lambda \in \mathbb{R}$ such that

$$y'(t) \leq \lambda y(t)$$

then we have that

$$y(t) \leq y(0)e^{\lambda t}.$$

Prove Grönwall's inequality.

Remark: The above inequality can be generalised to show that if

$$y'(t) \leq \lambda(t) y(t)$$

then $y(t) \leq y(0)e^{\int_0^t \lambda(s)ds}$. There are additional important Grönwall inequalities which we won't mention at this point.

Solution: Define

$$z(t) = e^{-\lambda t} y(t)$$

$$\frac{dz(t)}{dt} = e^{-\lambda t} y'(t) - \lambda e^{-\lambda t} y(t)$$

$$= e^{-\lambda t}\left[y'(t) - \lambda y(t)\right] \leq 0$$

$\Rightarrow$ $z(t)$ is non-increasing! $\forall t > 0$

$$z(t) \leq z(0)$$

or

$$e^{-\lambda t} y(t) \leq e^{0} y(0)$$

i.e

$$y(t) \leq e^{\lambda t} y(0)$$

Exercise 5. Let $T > 0$ be given and define

$$\Omega_T = (a, b) \times (0, T]$$

for a given $-\infty < a < b < \infty$.

(i) Show that there exists at most one solution $u \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ to the problem

$$\begin{cases} u_t - u_{xx} = 1 & (x,t) \in \Omega_T, \\ u = 0 & (x,t) \in \Gamma_T, \end{cases} \tag{1}$$

where $\Gamma_T = (a, b) \times \{0\} \cup \{a\} \times [0, T] \cup \{b\} \times [0, T]$.

(ii) Assume that u is a $C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ solution to (1). Show that for any $(x, t) \in \Omega_T$

$$0 \leq u(x, t) \leq t.$$

Solution:

(i) Assume u_1 and u_2 solve (1).
Then $w = u_1 - u_2$ solves

$$w_t - w_{xx} = 0 \qquad \Omega_T$$
$$w = 0 \qquad \text{on } \Gamma_T$$

$w \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat eq.
so by the weak min/max principles

$$\max_{\Omega_T} w = \max_{\Gamma_T} w = 0$$

$$\min_{\Omega_T} w = \min_{\Gamma_T} w = 0$$

$\Rightarrow w \equiv 0$.

(ii) $u_t - u_{xx} \geq 0$ on Ω_T
so u satisfies a weak min principle, i.e

$$\min_{\overline{\Omega_T}} u(x,t) = \min_{\Gamma_T} u(x,t) = 0$$

$$\Rightarrow u(x,t) \geq 0 \quad \forall x,t$$

The ineq. $u(x,t) \leq t$ is equivalent to $u(x,t) - t \leq 0$

Defining $v(x,t) = u(x,t) - t$ we find that

$$v_t - v_{xx} = u_t - 1 - u_{xx} = 0 \quad \text{in } \Omega$$

$$v(x,0) = u(x,0) - t = -t$$

v satisfies the heat eq and consequently by the max principle

$$\max_{\overline{\Omega_T}} v(x,t) = \max_{\Gamma_T} v(x,t) = \max_{\Gamma_T}(-t) = 0$$

i.e. $v(x,t) \leq 0 \quad \forall x,t$