**PARTIAL DIFFERENTIAL EQUATIONS III & V**
**PROBLEM CLASS 8 SOLUTION**

**Exercise 1** *(Properties of the fundamental solution to the Heat equation).* Let

$$\Phi(x,t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4kt}},$$

where $k > 0$ is given, be the fundamental solution to the Heat equation.

(i) Show that for any $p \geq 1$

$$\|\Phi(x,t)\|_{L^p(\mathbb{R}^n)} = \frac{(4\pi k t)^{\frac{n(1-p)}{2p}}}{p^{\frac{n}{2p}}}$$

You may use the fact that $\int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2}} dy = (2\pi)^{\frac{n}{2}}$.

In particular,

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = \|\Phi(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$$

for all $t > 0$.

(ii) Young's convolution inequality states that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $p, q \in [1,\infty]$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

for some $r \in [1,\infty]$ then $f * g \in L^r(\mathbb{R}^n)$ and

$$\left\|f * g\right\|_{L^r(\mathbb{R}^n)} \leq \left\|f\right\|_{L^p(\mathbb{R}^n)} \left\|g\right\|_{L^q(\mathbb{R}^n)}.$$

Use this to show that for any $g \in C_c(\mathbb{R}^n)$ we have that for any $r \geq 1$ and any $1 \leq p \leq r$

$$\left\|\Phi(\cdot,t) * g\right\|_{L^r} \leq \frac{C_{p,n,k}\left\|g\right\|_{L^{\frac{rp}{rp+p-r}}(\mathbb{R}^n)}}{t^{\frac{n(p-1)}{2p}}}$$

where $C_{p,n,k}$ is an explicit constant that depends only on $p$, $n$, and $k$.

*Solution:*

(i)

$$\|\Phi(x,t)\|^p_{L^p(\mathbb{R}^n)} = \frac{1}{(4\pi k t)^{\frac{pn}{2}}} \int_{\mathbb{R}^n} e^{-\frac{p|x|^2}{4kt}} dx$$

$$\underset{y=\sqrt{\frac{p}{2kt}}x}{=} \frac{1}{(4\pi k t)^{\frac{pn}{2}}} \left(\frac{2kt}{p}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2}} dy = \frac{(4\pi k t)^{\frac{n(1-p)}{2}}}{p^{\frac{n}{2}}},$$

from which the result follows.

(ii) This follows from the fact that if[1]

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

[1]Note that this inequality automatically implies that $p, q \leq r$ if $p, q, r \in [1,\infty]$.

then $q=\frac{rp}{rp+p-r}$, part (i), and Young's convolution inequality. Indeed,

$$\left\|\Phi(\cdot,t)*g\right\|_{L^r}\le\|\Phi\|_{L^p(\mathbb{R}^n)}\left\|g\right\|_{L^{\frac{rp}{rp+p-r}}(\mathbb{R}^n)}=\frac{(4\pi kt)^{\frac{n(p-1)}{2p}}}{p^{\frac{n}{2p}}}\left\|g\right\|_{L^{\frac{rp}{rp+p-r}}(\mathbb{R}^n)}.$$

**Exercise 2** *(Bonus – additional bounds on $u=\Phi*g$).* Consider the solution to the heat equation

$$\begin{aligned} &u_t(x,t)-k\Delta u(x,t)=0, && x\in\mathbb{R}^n,\ t>0,\\ &u(x,0)=g(x), && x\in\mathbb{R}^n, \end{aligned}$$

where $k>0$ and $g\in L^1(\mathbb{R}^n)$, given by

$$u(x,t)=\int_{\mathbb{R}^n}\Phi(x-y,t)g(y)dy$$

with

$$\Phi(x,t)=\frac{1}{(4\pi kt)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4kt}}.$$

(i) Show that if $g\in L^\infty(\mathbb{R}^n)$ then so is $u$. Moreover

$$\|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\le\left\|g\right\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) If $g\in L^2(\mathbb{R}^n)$ one can show that $u(\cdot,t)\in L^2(\mathbb{R}^n)$ for all $t>0$ (follows from the previous exercise!). Show that for any $t>0$

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)}\le\left\|g\right\|_{L^2(\mathbb{R}^n)}.$$

*Hint:* You may use the following:

$$\Phi\ge0,\qquad\text{and}\qquad\int_{\mathbb{R}^n}\Phi(z,t)dz=1,\ \forall t>0.$$

$$\widehat{\Phi}(\xi,t)=\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-kt|\xi|^2}.$$

*Solution:*

(i) We have that for any $x\in\mathbb{R}^n$ and $t>0$

$$|u(x,t)|\le\int_{\mathbb{R}^n}\left|\Phi(x-y,t)\right|\left|g(y)\right|dy\le\left\|g\right\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\Phi(x-y,t)dy$$

$$\underset{\substack{z=x-y\\ dz=\left|(-1)^n\right|dy}}{=}\left\|g\right\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\Phi(z,t)dz=\left\|g\right\|_{L^\infty(\mathbb{R}^n)}.$$

Consequently

$$\|u\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\le\left\|g\right\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) Passing to the Fourier transform in the spatial variable we find that

$$\widehat{u}(\xi,t)=\widehat{\Phi(\cdot,t)*g}(\xi)=(2\pi)^{\frac{n}{2}}\,\widehat{\Phi}(\xi,t)\,\widehat{g}(\xi)=\widehat{g}(\xi)\,e^{-kt|\xi|^2}.$$

Using to Plancherel's identity we find that

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}=\|\widehat{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}=\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi$$

$$=\int_{\mathbb{R}^n}\left|g(\xi)\right|^2e^{-2kt|\xi|^2}d\xi\le\int_{\mathbb{R}^n}\left|g(\xi)\right|^2d\xi=\left\|\widehat{g}\right\|^2_{L^2(\mathbb{R}^n)}=\left\|g\right\|^2_{L^2(\mathbb{R}^n)},$$

giving us the desired result.

**Exercise 3** *(The energy method: Uniqueness for the heat equation in a time dependent domain).* Let $k>0$, $T>0$ be given. Let $a, b : [0,T] \to \mathbb{R}$ be smooth functions such that $a(t) < b(t)$ for all $t \in [0,T]$. Let $U \subset \mathbb{R} \times (0,T]$ be the non-cylindrical domain

$$U = \{(x,t) \in \mathbb{R} \times (0,T] \mid a(t) < x < b(t)\}.$$

Consider the heat equation

$$\begin{cases} u_t - k u_{xx} = f(x,t) & (x,t) \in U, \\ u(a(t),t) = g_1(t) & t \in [0,T], \\ u(b(t),t) = g_2(t) & t \in [0,T], \\ u(x,0) = u_0(x) & x \in (a(0), b(0)). \end{cases}$$

Use the energy method to prove that the equation has at most one smooth solution.

*Solution:*
Assuming there exist two smooth solution to the equation, $u_1$ and $u_2$, we start by defining $w = u_1 - u_2$. The linearity of the equation implies that $w$ solves the equation

$$\begin{cases} w_t - k w_{xx} = 0 & (x,t) \in U, \\ w(a(t),t) = 0 & t \in [0,T], \\ w(b(t),t) = 0 & t \in [0,T], \\ w(x,0) = 0 & x \in (a(0), b(0)). \end{cases}$$

As is common with the energy method, we will multiply our equation by a function and integrating by parts. In this case (though not always!) it will be $w$. We have that

$$\int_{a(t)}^{b(t)} w_t(x,t)\, w(x,t)\, dx = k \int_{a(t)}^{b(t)} w(x,t)\, w_{xx}(x,t)\, dx.$$

Since

$$\int_{a(t)}^{b(t)} w(x,t)\, w_{xx}(x,t)\, dx = \underbrace{w(b(t),t)}_{=0} w_x(b(t),t) - \underbrace{w(a(t),t)}_{=0} w_x(a(t),t)$$
$$- \int_{a(t)}^{b(t)} w_x^2(x,t)\, dx = - \int_{a(t)}^{b(t)} w_x^2(x,t)\, dx$$

and $w_t w = \frac{1}{2}\partial_t\left(w^2\right)$ we find that

$$\int_{a(t)}^{b(t)} \partial_t\left(w^2\right)(x,t)\, dx = -2k \int_{a(t)}^{b(t)} w_x^2(x,t)\, dx.$$

We would like to write $\int_{a(t)}^{b(t)} \partial_t\left(w^2\right) dx$ as a full time derivative. This is not immediately clear as the boundaries of the integration also depend on $t$. We notice, however, that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} w^2(x,t)\, dx = \int_{a(t)}^{b(t)} \partial_t\left(w^2\right)(x,t)\, dx + \underbrace{w^2(b(t),t)}_{=0} b'(t)$$
$$- \underbrace{w^2(a(t),t)}_{=0} a'(t) = \int_{a(t)}^{b(t)} \partial_t\left(w^2\right)(x,t)\, dx,$$

which is justified since all the functions are smooth. We conclude that our equation can be written as

$$\frac{d}{dt}\int_{a(t)}^{b(t)} w^2(x,t)\,dx = -2k\int_{a(t)}^{b(t)} w_x^2(x,t)\,dx \le 0,$$

which implies that the energy

$$E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx$$

is non-increasing. Consequently

$$0 \le E(t) \le E(0) = \int_{a(0)}^{b(0)} w^2(x,0)\,dx = 0$$

from which we conclude that, since $w$ is continuous, $w(x,t) = 0$ on $U$ or $u_1 \equiv u_2$.

**Remark:** We could have started by guessing (or being given) the energy and finding its derivative. Indeed, defining

$$E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx$$

we find that (again, all the functions are smooth)

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} \partial_t\left(w^2\right)(x,t)\,dx + \underbrace{\frac{w^2(b(t),t)\,b'(t)}{2}}_{=0}$$

$$-\underbrace{\frac{w^2(a(t),t)\,a'(t)}{2}}_{=0} = \int_{a(t)}^{b(t)} w(x,t)\,w_t(x,t)\,dx$$

$$= k\int_{a(t)}^{b(t)} w(x,t)\,w_{xx}(x,t)\,dx = k\underbrace{w(b(t),t)}_{=0}\,w_x(b(t),t) - k\underbrace{w(a(t),t)}_{=0}\,w_x(a(t),t)$$

$$-k\int_{a(t)}^{b(t)} w_x^2(x,t)\,dx = -k\int_{a(t)}^{b(t)} w_x^2(x,t)\,dx \le 0.$$

**Exercise 4** *(Grönwall's inequality).* Another important inequality in the study of PDEs, in particular in the study of long time behaviour of solutions, is the so-called Grönwall's inequality which state that if $y : [0,T] \to \mathbb{R}$ is continuous and differentiable on $(0,T)$, and if there exists $\lambda \in \mathbb{R}$ such that

$$y'(t) \le \lambda y(t)$$

then we have that

$$y(t) \le y(0)e^{\lambda t}.$$

Prove Grönwall's inequality.

**Remark:** The above inequality can be generalised to show that if

$$y'(t) \le \lambda(t)y(t)$$

then $y(t) \le y(0)e^{\int_0^t \lambda(s)ds}$. There are additional important Grönwall inequalities which we won't mention at this point.

*Solution:*
We note that we can write our inequality as

$$y'(t) - \lambda y(t) \leq 0.$$

Had we had equality, we would have used the integrating factor $e^{\int(-\lambda)dt} = e^{-\lambda t}$. This motivates us to define $z(t) = e^{-\lambda t}y(t)$. We find that

$$z'(t) = e^{-\lambda t}\left(y'(t) - \lambda y(t)\right) \leq 0.$$

Since $z(t)$ is differentiable on $(0, T)$ and continuous on $[0, T]$ (as $y(t)$ is) we conclude that $z(t)$ must be non-increasing on $[0, T]$. Consequently, for any $t \in [0, T]$ we have that $z(t) \leq z(0) = y(0)$ which implies

$$y(t) = z(t)e^{\lambda t} \leq y(0)e^{\lambda t}.$$

**Exercise 5**. Let $T > 0$ be given and define

$$\Omega_T = (a, b) \times (0, T]$$

for a given $-\infty < a < b < \infty$.

(i) Show that there exists at most one solution $u \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ to the problem

$$\begin{cases} u_t - u_{xx} = 1 & (x, t) \in \Omega_T, \\ u = 0 & (x, t) \in \Gamma_T, \end{cases} \tag{1}$$

where $\Gamma_T = (a, b) \times \{0\} \cup \{a\} \times [0, T] \cup \{b\} \times [0, T]$.

(ii) Assume that $u$ is a $C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ solution to (1). Show that for any $(x, t) \in \Omega_T$

$$0 \leq u(x, t) \leq t.$$

*Solution:*

(i) Assuming that there are two $C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ solutions, $u_1$ and $u_2$, we define $w = u_1 - u_2$ which is also a function in $C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ and satisfies the equation

$$\begin{cases} w_t - w_{xx} = 0 & (x, t) \in \Omega_T, \\ u = 0 & (x, t) \in \Gamma_T, \end{cases}$$

Using the weak maximum and weak minimum principles for the heat equation we find that

$$\max_{\overline{\Omega_T}} w(x, t) = \max_{\Gamma_T} w(x, t) = 0$$

and

$$\min_{\overline{\Omega_T}} w(x, t) = \min_{\Gamma_T} w(x, t) = 0$$

which implies that $w \equiv 0$ on $\overline{\Omega_T}$, or equivalently that $u_1 \equiv u_2$.

(ii) We see that $u$ satisfies

$$u_t - u_{xx} = 1 > 0$$

and consequently, according to the weak minimum principle,

$$\min_{\overline{\Omega_T}} u(x, t) = \min_{\Gamma_T} u(x, t) = 0$$

showing that $u(x, t) \geq 0$ on $\Omega_T$.

We can't use the weak maximum principle on $u$ but we notice that the second inequality that we'd like to show, $u(x,t) \leq t$, can be rewritten as $u(x,t) - t \leq 0$. This motivates us to define $w(x,t) = u(x,t) - t$. We find that $w \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega_T}\right)$ and it satisfies

$$w_t - w_{xx} = u_t - 1 - u_{xx} = 0$$

in $\Omega_T$ and

$$w = 0 - t = -t$$

on $\Gamma_T$. Using the weak maximum principle principle for $w$ we find that

$$\max_{\overline{\Omega_T}} w(x,t) = \max_{\Gamma_T} w(x,t) = \max_{\Gamma_T}(-t) \leq 0$$

which implies that for any $x \in \Omega_T$

$$u(x,t) = w(x,t) + t \leq t.$$