Partial Differential Equations III/IV
Exercise Sheet 4

1. *Green's functions.* Consider Poisson's equation in one dimension with mixed Dirichlet-Neumann boundary conditions:

$$-u''(x) = f(x), \quad x \in (0,1),$$
$$u'(0) = 0, \; u(1) = 0,$$

where $f \in C([0,1])$. Find the unique solution of this equation. Write your solution in the form

$$u(x) = \int_0^1 G(x,y) f(y)\, dy.$$

2. *Homogenization.* Let $\alpha > 0$ be constant and $a : \mathbb{R} \to [\alpha, \infty)$ be 1–periodic, i.e., $a(x+1) = a(x)$ for all $x \in \mathbb{R}$. For $\varepsilon > 0$, define the ε–periodic function $a_\varepsilon(x) = a(\frac{x}{\varepsilon})$. Consider the steady diffusion equation

$$-(a_\varepsilon(x) u_\varepsilon'(x))' = f(x), \quad x \in (0,1), \tag{1}$$
$$u_\varepsilon(0) = u_\varepsilon(1) = 0, \tag{2}$$

where $f : [0,1] \to \mathbb{R}$ is continuous. This models the equilibrium temperature distribution in a metal bar with heat source f, where the bar is made of a *nonhomogeneous* material. The thermal conductivity a_ε depends on position and represents a metal bar composed of repeating segments of length ε. We will discover that, for small ε, the bar behaves as if it were made of a *homogenous* material with constant thermal conductivity a_0. You might guess that a_0 is some sort of average of a_ε, but what is the correct notion of average? This is called a *homogenization* problem. Homogenization is an active research area.

(i) Derive the following solution to (1), (2):

$$u_\varepsilon(x) = \int_0^1 \frac{1}{a_\varepsilon(z)} \int_0^z f(y)\, dy\, dz \int_0^x \frac{1}{a_\varepsilon(z)}\, dz \left(\int_0^1 \frac{1}{a_\varepsilon(z)}\, dz \right)^{-1} - \int_0^x \frac{1}{a_\varepsilon(z)} \int_0^z f(y) dy\, dz.$$

(ii) Let $\varepsilon_n = \frac{1}{n}$. Prove that $\lim_{n\to\infty} u_{\varepsilon_n}(x) = u_0(x)$, where

$$u_0(x) = x \overline{\left(\frac{1}{a}\right)} \int_0^1 \int_0^z f(y)\, dy dz - \overline{\left(\frac{1}{a}\right)} \int_0^x \int_0^z f(y)\, dy dz$$

and where

$$\overline{\left(\frac{1}{a}\right)} = \int_0^1 \frac{1}{a(y)}\, dy$$

is the average of $1/a$ over one period.

Hint: Use the following deep result, which you do not need to prove: Let $g \in L^\infty(\mathbb{R})$ be 1–periodic. For any interval $[c,d] \subseteq \mathbb{R}$,

$$\lim_{n\to\infty} \int_c^d g(nz) h(z)\, dz = \int_c^d \overline{g}\, h(z)\, dz \qquad \forall\, h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R}) \tag{3}$$

where $\overline{g}$ is the average of g over one period, i.e., $\overline{g} = \int_0^1 g(y)\, dy$. This also holds under the weaker assumption that $h \in L^1(\mathbb{R})$. Define $g_n(y) := g(ny)$. We say that g_n converges weak-$*$ in $L^\infty(\mathbb{R})$ to $\overline{g}$ as $n \to \infty$, but that's another story for a functional analysis course.

(iii) Write u_0 in the form

$$u_0(x) = \int_0^1 G(x,y) f(y)\, dy$$

for some symmetric Green's function G, which you should determine.

(iv) Use part (ii) to show that u_0 satisfies the steady diffusion equation

$$\begin{aligned} -a_0 u_0''(x) &= f(x), \quad x \in (0,1), \\ u_0(0) &= u_0(1) = 0, \end{aligned}$$

where the thermal conductivity a_0 is the constant

$$a_0 = \frac{1}{\overline{\left(\frac{1}{a}\right)}}.$$

(v) Observe that a_0 is the reciprocal of the average of the reciprocal of a. In general this is not the same as the average of a, as we now illustrate. Let

$$a(x) = \begin{cases} \frac{1}{2} & x \in (0, \frac{1}{2}), \\ 1 & x \in (\frac{1}{2}, 1), \end{cases}$$

and extend a by periodicity to the real line. This represents a composite metal bar composed of segments of two homogeneous materials with different thermal conductivities. Compute a_0 and $\overline{a}$. Verify that $a_0 \neq \overline{a}$.

(vi) Bonus, optional question (hard): Prove (3). This is a form of the *Riemann-Lebesgue Lemma*. It is a generalisation of the Riemann-Lebesgue Lemma for Fourier series and the Fourier transform.

Hint: Start by writing

$$\int_c^d g(nz) h(z)\, dz = \int_c^d \left(\frac{1}{n} \int_0^{nz} g(y)\, dy \right)_z h(z)\, dz.$$

3. *Radial symmetry of Laplace's equation on* $\mathbb{R}^n$. Let

$$O(n, \mathbb{R}) = \{ M \in \mathbb{R}^{n \times n} : M M^T = M^T M = I \}$$

be the set of real, n–by–n orthogonal matrices, which represent rotations and reflections of $\mathbb{R}^n$. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function and let $R \in O(n, \mathbb{R})$. Define $w : \mathbb{R}^n \to \mathbb{R}$ by $w(\boldsymbol{x}) := v(R\boldsymbol{x})$. Prove that w is also harmonic.

4. *Fundamental solution of Poisson's equation in 3D.* Let Φ be the fundamental solution of Poisson's equation in $\mathbb{R}^3$:

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi} \frac{1}{|\boldsymbol{x}|}.$$

(i) Let $R > 0$. Compute $\|\Phi\|_{L^1(B_R(\mathbf{0}))}$.

Hint: Use spherical polar coordinates or, simpler, the following formula:

$$\int_{B_R(\mathbf{0})} f(\boldsymbol{x})\, d\boldsymbol{x} = \int_0^R \left(\int_{\partial B_r(\mathbf{0})} f(\boldsymbol{y})\, dS(\boldsymbol{y}) \right) dr.$$

(ii) Prove that $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

(iii) Prove that $\Phi \notin L^1(\mathbb{R}^3)$.

(iv) Prove that $\nabla \Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

5. *Fundamental solution of Poisson's equation in 1D.* Let $f \in C_c^2(\mathbb{R})$ be twice continuously differentiable with compact support. Define

$$\Phi(x) := \begin{cases} x & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

We call Φ the fundamental solution of Poisson's equation in $\mathbb{R}$. Define $u := \Phi * f$. Prove that u satisfies

$$-u''(x) = f(x), \quad x \in \mathbb{R}.$$

Hint: Write

$$u''(x) = (\Phi * f)''(x) = (f * \Phi)''(x) = \int_{-\infty}^{\infty} f''(x-y)\Phi(y)\, dy.$$

Now integrate by parts. Unlike for the case of Poisson's equation in $\mathbb{R}^n$, $n \geq 2$, you do not need to remove a ball of radius ε around the origin since Φ does not have a singularity at the origin in 1D. In fact, unlike in higher dimensions, Φ is continuous in 1D.

6. *The function spaces L^1 and L^1_{loc}.* Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^k$. Let $R > 0$. Find all the values of $k \in \mathbb{R}$ for which

   (i) $f \in L^1((-R, R))$,

   (ii) $f \in L^1((R, \infty))$,

   (iii) $f \in L^1_{\text{loc}}(\mathbb{R})$,

   (iv) $f \in L^1(\mathbb{R})$.

7. *Properties of the convolution.* Let $\varphi \in L^1_{\text{loc}}(\mathbb{R})$, $f \in C_c(\mathbb{R})$. Prove

   (i) $|(\varphi * f)(x)| < \infty$ for all $x \in \mathbb{R}$;

   (ii) if $\varphi \in L^1(\mathbb{R})$, then $\varphi * f \in L^\infty(\mathbb{R})$;

   (iii) the convolution is commutative: $\varphi * f = f * \varphi$.

   Remark: It can be shown that $\varphi * f \in L^1(\mathbb{R})$ if $\varphi, f \in L^1(\mathbb{R})$. Consequently $(L^1(\mathbb{R}), *)$ is an algebra.

8. *The Poincaré inequality for functions that vanish on the boundary.* Prove that there exists a constant $C > 0$ such that

$$\int_a^b |f(x)|^2\, dx \leq C \int_a^b |f'(x)|^2\, dx$$

   for all $f \in C^1([a, b])$ satisfying $f(a) = f(b) = 0$.
   Hint: The proof is similar to, and simpler than, the version we proved in Section 4.3.

9. *The Poincaré inequality on unbounded domains.*

   (i) Construct a sequence $f_n \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $f_n' \in L^2(\mathbb{R})$ and

$$\|f_n'\|_{L^2(\mathbb{R})} = \text{constant}, \qquad \|f_n\|_{L^2(\mathbb{R})} \overset{n\to\infty}{\longrightarrow} \infty.$$

   This means that the Poincaré inequality does not hold on $\mathbb{R}$, i.e, there does not exists any $C > 0$ such that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \leq C \int_{-\infty}^{\infty} |f'(x)|^2\, dx$$

   for all $f \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $f' \in L^2(\mathbb{R})$.

   (ii) Let Ω be the unbounded domain $\Omega = (a, b) \times (-\infty, \infty)$. Prove that there exists $C > 0$ such that

$$\int_\Omega |f(\boldsymbol{x})|^2 \, d\boldsymbol{x} \le C \int_\Omega |\nabla f(\boldsymbol{x})|^2 \, d\boldsymbol{x}$$

for all $f \in C^1(\overline{\Omega}) \cap L^2(\Omega)$ with $\nabla f \in L^2(\Omega)$ and with $f(a, y) = f(b, y) = 0$ for all $y \in \mathbb{R}$. More generally, the Poincaré inequality is true if Ω is bounded between two parallel hyperplanes (lines in 2D, planes in 3D, etc). In this example Ω is bounded between the lines $x = a$ and $x = b$.

10. *The Poincaré constant depends on the domain.* Let $C_1 > 0$ satisfy

$$\int_0^1 |f(x)|^2 \, dx \le C_1 \int_0^1 |f'(x)|^2 \, dx \tag{4}$$

for all $f \in C^1([0, 1])$ with $f(0) = f(1) = 0$. By using a change of variables, use (4) to prove that

$$\int_0^L |g(x)|^2 \, dx \le C_L \int_0^L |g'(x)|^2 \, dx \tag{5}$$

for all $g \in C^1([0, L])$ with $g(0) = g(L) = 0$, where

$$C_L = L^2 C_1.$$

Remark: Those with a good physical intuition will see that C_L must have units of length squared, otherwise the units in equation (5) do not match: if g is dimensionless, then $\int_0^L |g|^2 \, dx$ has units of length whereas $\int_0^L |g'|^2 \, dx$ has units of 1/length.

11. *Eigenvalues of $-\Delta$: Can you hear the shape of a drum?* Let $\Omega \subset \mathbb{R}^2$ be open and bounded with smooth boundary. Let $u : \overline{\Omega} \to \mathbb{C}$ be a smooth eigenfunction of $-\Delta$ that vanishes on $\partial\Omega$, which means $u \ne 0$ and

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

for some $\lambda \in \mathbb{C}$. We say that λ is the eigenvalue associated to the eigenfunction u. Use the energy method to prove that λ is real and that $\lambda > 0$.

Hint: Start by multiplying the PDE by u^*, the complex conjugate of u.

Remark: This eigenvalue problem arises if you seek a solution of the form $v(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{i\omega t}$ of the wave equation $v_{tt} = c\Delta v$ with clamped boundary conditions, which models small vibrations of a drum of shape Ω. The eigenvalues λ are related to the principal frequencies ω of the drum by $\lambda = \omega^2/c$. It can be shown that there are countably-many eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$. Moreover, H. Weyl showed that the eigenvalues determine the area of Ω; you can hear the area of a drum. In 1966 M. Kac asked whether the eigenvalues determine the shape of Ω; can you hear the shape of a drum? This was disproved in 1992 by Gordon, Webb and Wolpert, who constructed two distinct, non-convex polygons with the same principal frequencies. As a final twist, it is possible to hear the shape of a *convex* drum; two distinct convex sets have different principal frequencies.

12. *The optimal Poincaré constant and eigenvalues of $-\Delta$.*

    (i) Use the energy method to show that if the pair $(\lambda, u) \in \mathbb{R} \times C^2(\overline{\Omega})$, $u \ne 0$, satisfies

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

    then

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}}. \tag{7}$$

This is the *Rayleigh quotient formula* for the eigenvalue λ in terms of the eigenfunction u. It can be shown that there are countably many eigenvalues and that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$.
Remark: The Rayleigh quotient formula for the matrix eigenvalue problem $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is

$$\lambda = \frac{\boldsymbol{x}^T A\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}.$$

This has the same form of (7) but with the L^2–inner product in (7) replaced by the dot product.

(ii) Define

$$V = \{\varphi \in C^1(\overline{\Omega}) \, : \, \varphi = 0 \text{ on } \partial\Omega, \ \varphi \neq 0\}$$

and define the functional $E : V \to \mathbb{R}$ by

$$E[v] = \frac{\displaystyle\int_\Omega |\nabla v|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |v|^2 \, d\boldsymbol{x}}.$$

Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E, i.e.,

$$E[u] = \min_{v \in V} E[v].$$

Prove that

$$\begin{aligned} -\Delta u &= \lambda_1 u \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where λ_1 is the smallest eigenvalue of (6), and that

$$E[u] = \lambda_1.$$

Remark: We have shown that the minimum value of the Rayleigh quotient E is the smallest eigenvalue of the operator $-\Delta$ on V, and that E is minimised by the corresponding eigenfunction. This is analogous to the result that if A is a symmetric positive definite matrix, then the minimum value of the Rayleigh quotient $\boldsymbol{x}^T A\boldsymbol{x}/\boldsymbol{x}^T\boldsymbol{x}$ is the smallest eigenvalue of A, and it is minimised by the corresponding eigenvector. (Recall also that the maximum value of the Rayleigh quotient is the largest eigenvalue of A). Equivalently, the minimum value of the quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ over the sphere $|\boldsymbol{x}| = 1$ is the smallest eigenvalue of A.

(iii) The *optimal Poincaré constant* is the smallest value of $C > 0$ such that

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)} \tag{8}$$

for all $f \in C^1(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$. Let us denote this value of C by C_{P}. Show that

$$\frac{1}{C_{\mathrm{P}}} = \inf_{f \in V} \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}.$$

(iv) Combine parts (ii) and (iii) to conclude that the optimal Poincaré constant is

$$C_{\mathrm{P}} = \frac{1}{\sqrt{\lambda_1}}.$$

(v) Use part (iv) to show that the optimal Poincaré constant for the domain $\Omega = (0, 2\pi)$ is $C_{\mathrm{P}} = 2$. Compare this with the constant you obtained in Q8 with $a = 0$, $b = 2\pi$.
Remark: The optimal constant $C_{\mathrm{P}} = 2$ can also be obtained using Fourier series.

13. *Uniqueness for Poisson's equation with Robin boundary conditions.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $\alpha > 0$. Use the energy method to show that there is at most one smooth solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} + \alpha u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where $\boldsymbol{n}$ denotes the outward-pointing unit normal vector field to $\partial\Omega$. This type of boundary condition is called a *Robin* boundary condition.

14. *Uniqueness and stability for a more general elliptic problem.* Consider the linear, second-order, elliptic PDE

$$\begin{aligned} -\operatorname{div}(A\,\nabla u) + \boldsymbol{b} \cdot \nabla u + cu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{9}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with smooth boundary, $A \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$, $\boldsymbol{b} \in C(\overline{\Omega}; \mathbb{R}^n)$, and $c, f, g \in C(\overline{\Omega})$. Assume that c is nonnegative, $\operatorname{div} \boldsymbol{b} = 0$, and A is uniformly positive definite, i.e., there exists a constant $\alpha > 0$ such that $\boldsymbol{y}^T A(\boldsymbol{x})\boldsymbol{y} \geq \alpha|\boldsymbol{y}|^2$ for all $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$.

(i) Prove that (9) has at most one solution $u \in C^2(\overline{\Omega})$.

(ii) Let $(A_n)_n$ be a sequence of matrix valued functions satisfying the previous conditions (in particular they are all uniformly positive definite with the same $\alpha > 0$). The rest of the data is fixed and satisfy all the previous assumptions. Suppose that u_n is the unique solutions to

$$\begin{aligned} -\operatorname{div}(A_n\,\nabla u_n) + \boldsymbol{b} \cdot \nabla u_n + cu_n &= f \quad \text{in } \Omega, \\ u_n &= g \quad \text{on } \partial\Omega, \end{aligned}$$

Show that if $A_n \to A$ uniformly in $\overline{\Omega}$ as $n \to +\infty$ and if the problem with the limit matrix A has a unique solution u, then $\nabla u_n \to \nabla u$ in $L^2(\Omega)$, as $n \to +\infty$.

Remarks: Uniqueness may fail if c is negative; see the PDEs exam from May 2017, Q4(b). Another obstacle to uniqueness is unbounded domains; see Exercise Sheet 5. We say that the PDE (9) has *divergence form*, which is the most convenient form for energy methods (compare (9) with the general form of elliptic PDEs given in Definition 4.1).

15. *Uniqueness for a degenerate diffusion equation.* Let $m > 1$ be a constant. Show that the following steady degenerate diffusion equation has a unique positive solution:

$$\begin{aligned} \Delta u^m &= 0 \quad \text{in } \Omega, \\ u &= \pi \quad \text{on } \partial\Omega. \end{aligned}$$

Remark: Observe that $\Delta u^m = \operatorname{div} \nabla(u^m) = \operatorname{div}(mu^{m-1}\nabla u) = \operatorname{div}(a(u)\nabla u)$ with $a(u) = mu^{m-1}$. We call the equation $\Delta u^m = 0$ the *degenerate* diffusion equation since the diffusion coefficient $a(u) = mu^{m-1}$ vanishes when $u = 0$.

16. *The H_0^1 and H^1 norms.* Let

$$V = \{f \in C^1([a, b]) : f(a) = f(b) = 0\}.$$

(i) Prove that $\|\cdot\|_{L^2([a,b])}$ is a norm on $C([a,b])$.
Hint: The only difficulty is proving the triangle inequality. Write

$$\|f+g\|^2_{L^2([a,b])} = \|f\|^2_{L^2([a,b])} + \|g\|^2_{L^2([a,b])} + 2\int_a^b f(x)g(x)\,dx$$

and use the Cauchy-Schwarz inequality.

(ii) Prove that $\|\cdot\|_{H^1([a,b])}$ is a norm on $C^1([a,b])$.

(iii) Prove that $\|\cdot\|_{H^1_0([a,b])}$ is a norm on V. Is it a norm on $C^1([a,b])$?

(iv) Prove that the norms $\|\cdot\|_{H^1([a,b])}$ and $\|\cdot\|_{H^1_0([a,b])}$ are equivalent on V, which means that there exist constants $c, C > 0$ such that

$$c\|f\|_{H^1_0([a,b])} \le \|f\|_{H^1([a,b])} \le C\|f\|_{H^1_0([a,b])} \quad \forall\, f \in V.$$

Hint: Use the Poincaré inequality to find C.
Remark: If two norms are equivalent, then a sequence converges in one norm if and only if it converges in the other.

17. *Continuous dependence.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $u \in C^2(\overline{\Omega})$ satisfy

$$\begin{aligned} -\mathrm{div}(A\,\nabla u) + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in C(\overline{\Omega})$, $A \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$ is uniformly elliptic (see Q9), and $c > 0$ is a constant. Prove that there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

18. *Continuous dependence with a first-order term.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $k > 0$, $c > 0$ be constants and let $f : \overline{\Omega} \to \mathbb{R}$, $\boldsymbol{b} : \overline{\Omega} \to \mathbb{R}^n$ be continuous. Let $u \in C^2(\overline{\Omega})$ satisfy

$$\begin{aligned} -k\Delta u + \boldsymbol{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

(a) Prove that

$$k\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x} + c\|u\|^2_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

(b) Prove that for all $\varepsilon > 0$

$$\left|\int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x}\right| \le \|\boldsymbol{b}\|_{L^\infty(\Omega)} \left(\varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)}\right).$$

Hint: You may use the Young inequality, which states that

$$\alpha\beta \le \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 \quad \forall\, \alpha, \beta > 0.$$

(c) Prove that for all $\varepsilon > 0$

$$\left(k - \varepsilon\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)\|\nabla u\|^2_{L^2(\Omega)} + \left(c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon}\right)\|u\|^2_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

(d) Find a constant $c_0 > 0$ such that if $c > c_0$, then

$$\|u\|_{H^1(\Omega)} \le M\|f\|_{L^2(\Omega)}$$

for some constant $M > 0$.

(e) Show that if $c > c_0$, then u is the only solution of (10).

19. *Neumann boundary conditions for variational problems.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Define $E : C^1(\overline{\Omega}) \to \mathbb{R}$ by

$$E[v] = \frac{1}{2}\int_\Omega |\nabla v|^2 \, d\boldsymbol{x} - \int_\Omega f v \, d\boldsymbol{x}.$$

Suppose that $u \in C^1(\overline{\Omega})$ minimises E:

$$E[u] = \min_{v \in C^1(\overline{\Omega})} E[v].$$

(i) Show that

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\boldsymbol{x} = \int_\Omega f\varphi \, d\boldsymbol{x} \quad \text{for all } \varphi \in C^1(\overline{\Omega}).$$

This is the weak formulation of Poisson's equation with zero Neumann boundary conditions, as the following part demonstrates:

(ii) Show that, if in addition $u \in C^2(\overline{\Omega})$, then

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\boldsymbol{n}$ denotes the outward-pointing unit normal vector field to $\partial\Omega$.

Hint: First choose test functions $\varphi \in C^1(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$. Use this to establish that $-\Delta u = f$. Then choose any test function $\varphi \in C^1(\overline{\Omega})$ and show that the boundary condition holds.

Remark: Observe that the Neumann boundary condition arises naturally without including it in the domain of E (cf. the case of Dirichlet boundary conditions in Section 4.5, where the boundary condition is included in the domain of the energy functional). Consequently Neumann boundary conditions are sometimes referred to as *natural* boundary conditions.

20. *The p–Laplacian operator.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let

$$V = \{\varphi \in C^1(\overline{\Omega}) \, : \, \varphi = 0 \text{ on } \partial\Omega\}.$$

For $1 \le p < \infty$, define $E_p : V \to \mathbb{R}$ by

$$E_p[v] = \frac{1}{p}\int_\Omega |\nabla v|^p \, d\boldsymbol{x} - \int_\Omega f v \, d\boldsymbol{x}.$$

We met the case $p = 2$ in Section 4.5 and the previous question. Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E_p:

$$E_p[u] = \min_{v \in V} E_p[v].$$

(i) Prove that u satisfies the PDE

$$\begin{aligned} -\Delta_p u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Δ_p is the p–Laplacian operator, which is defined by $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$. By taking $p = 2$ we recover the regular Laplacian operator: $\Delta_2 = \Delta$.

(ii) Show that

$$E_p[u] = \frac{1-p}{p}\int_\Omega |\nabla u|^p \, d\boldsymbol{x} = \frac{1-p}{p}\int_\Omega f u \, d\boldsymbol{x}.$$

21. *The minimal surface equation: PDEs and soap films.* This question is adapted from the PDEs exam, May 2017. Let $\Omega \subset \mathbb{R}^2$ be open and bounded with smooth boundary. Let $g : \partial\Omega \to \mathbb{R}$ be a given smooth function and let

$$V = \{\varphi \in C^1(\overline{\Omega}) \, : \, \varphi = g \text{ on } \partial\Omega\}.$$

Define $A : V \to \mathbb{R}$ by

$$A[v] = \int_\Omega \sqrt{1 + |\nabla v|^2} \, d\boldsymbol{x}.$$

Observe that $A[v]$ is the area of the surface $\{(x, y, v(x, y)) : (x, y) \in \Omega\}$, i.e., $A[v]$ is the surface area of the graph of v. Suppose that the graph of $u \in C^2(\overline{\Omega}) \cap V$ has minimal surface area amongst all graphs with given boundary g:

$$A[u] = \min_{v \in V} A[v].$$

Show that u satisfies the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{in } \Omega.$$

22. *Homogenization and the calculus of variations.* In this question we revisit the homogenization problem from Q2 from the viewpoint of the calculus of variations. Let $l, r \in \mathbb{R}$ be constants. Define

$$V = \{\varphi \in C^1([0, 1]) \, : \, \varphi(0) = l, \ \varphi(1) = r\}.$$

Let $\alpha > 0$ be constant and $a : \mathbb{R} \to [\alpha, \infty)$ be continuously differentiable. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Define the energy functional $E : V \to \mathbb{R}$ by

$$E[v] = \frac{1}{2}\int_0^1 a(x)|v'(x)|^2 \, dx - \int_0^1 f(x)v(x) \, dx.$$

Observe that we recover the Dirichlet energy when $a = 1$.

(i) Suppose that $u \in C^2([0, 1]) \cap V$ minimises E:

$$E[u] = \min_{v \in V} E[v].$$

Show that u satisfies

$$\begin{aligned} -(a(x)u'(x))' &= f(x), \quad x \in (0, 1), \\ u(0) = l, &\quad u(1) = r. \end{aligned}$$

(ii) Now assume that a is 1–periodic and consider the energy

$$E_n[v] = \frac{1}{2}\int_0^1 a(nx)|v'(x)|^2 \, dx - \int_0^1 f(x)v(x) \, dx.$$

Show that

$$\lim_{n \to \infty} E_n[v] = E_\infty[v] := \frac{1}{2}\int_0^1 \overline{a}\, |v'(x)|^2 \, dx - \int_0^1 f(x)v(x) \, dx$$

where

$$\overline{a} = \int_0^1 a(x) \, dx.$$

This means the nonhomogeneous energy E_n converges pointwise to the homogeneous energy E_∞.

(iii) Let $l = r = 0$. By part (i), if $u_n \in C^2([0,1]) \cap V$ is the minimiser of E_n, then u_n satisfies (1), (2) for $\varepsilon = 1/n$. Let $u_\infty \in C^2([0,1]) \cap V$ be the minimiser of E_∞. Show that

$$\lim_{n\to\infty} u_n(x) \neq u_\infty(x).$$

Interpretation: We have shown that E_n converges pointwise to E_∞, but the minimiser of E_n does not converge to the minimiser of E_∞. The moral of the story is that pointwise convergence is not the 'correct' notion of convergence when considering energy functionals. The correct notion is something called Γ–*convergence*. It can be shown that E_n Γ–converges in a suitable sense to

$$E_0[v] = \frac{1}{2}\int_0^1 a_0 |v'(x)|^2 \, dx - \int_0^1 f(x)v(x)\, dx$$

where a_0 was defined in Q2(iv). It is easy to check that the minimiser of E_n converges to the minimiser of E_0. The subject of Γ–convergence goes beyond the scope of this course, but the important property of Γ–convergence is that minimisers converge to minimisers. This means that if you are modelling a system with an energy functional and you want to simplify the functional by sending a large parameter $n \to \infty$ or a small parameter $\varepsilon \to 0$, then you should compute the Γ–limit, not the pointwise limit.