Partial Differential Equations III/IV
Exercise Sheet 4: Solutions

1. *Green's functions.* By the Fundamental Theorem of Calculus, integrating $u''(y) = f(y)$ over $[0, z]$, for any $z \in [0, 1]$, gives

$$\int_0^z u''(y)\,dy = -\int_0^z f(y)\,dy \quad \Longleftrightarrow \quad u'(z) = u'(0) - \int_0^z f(y)\,dy = -\int_0^z f(y)\,dy,$$

where we have used the boundary condition $u'(0) = 0$. Integrating again, this time over $[0, x]$, gives

$$\int_0^x u'(z)\,dz = -\int_0^x \int_0^z f(y)\,dy\,dz \quad \Longleftrightarrow \quad u(x) = u(0) - \int_0^x \int_0^z f(y)\,dydz.$$

Taking $x = 1$ and using the boundary condition $u(1) = 0$ yields

$$0 = u(0) - \int_0^1 \int_0^z f(y)\,dydz \quad \Longleftrightarrow \quad u(0) = \int_0^1 \int_0^z f(y)\,dydz.$$

Therefore

$$u(x) = \int_0^1 \int_0^z f(y)\,dydz - \int_0^x \int_0^z f(y)\,dydz.$$

By interchanging the order of integration we can write this as

$$\begin{aligned} u(x) &= \int_0^1 \int_y^1 f(y)\,dzdy - \int_0^x \int_y^x f(y)\,dzdy \\ &= \int_0^1 (1-y)f(y)\,dy - \int_0^x (x-y)f(y)\,dy \\ &= \int_0^x (1-y)f(y)\,dy + \int_x^1 (1-y)f(y)\,dy - \int_0^x (x-y)f(y)\,dy \\ &= \int_0^x (1-x)f(y)\,dy + \int_x^1 (1-y)f(y)\,dy. \end{aligned}$$

Therefore

$$u(x) = \int_0^1 G(x,y)f(y)\,dy$$

with

$$\boxed{G(x,y) = \begin{cases} 1-x & \text{if } y \le x, \\ 1-y & \text{if } y \ge x. \end{cases}}$$

2. *Homogenization.*

   (i) Integrate $(a_\varepsilon(y)u_\varepsilon(y))' = -f(y)$ over $y \in [0, z]$:

$$\begin{aligned} \int_0^z (a_\varepsilon(y)u_\varepsilon'(y))'\,dy = -\int_0^z f(y)\,dy \quad &\Longleftrightarrow \quad a_\varepsilon(z)u_\varepsilon'(z) - a_\varepsilon(0)u_\varepsilon'(0) = -\int_0^z f(y)\,dy \\ &\Longleftrightarrow \quad u_\varepsilon'(z) = \frac{a_\varepsilon(0)u_\varepsilon'(0)}{a_\varepsilon(z)} - \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy. \end{aligned}$$

Now integrate over $z \in [0, x]$:

$$\int_0^x u_\varepsilon'(z)\,dz = \int_0^x \left[\frac{a_\varepsilon(0)u_\varepsilon'(0)}{a_\varepsilon(z)} - \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\right] dz \iff$$

$$u_\varepsilon(x) = \underbrace{u_\varepsilon(0)}_{=0} + a_\varepsilon(0)u_\varepsilon'(0)\int_0^x \frac{1}{a_\varepsilon(z)}\,dz - \int_0^x \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\,dz. \tag{1}$$

We determine $u_\varepsilon'(0)$ by evaluating this expression at $x = 1$:

$$\underbrace{u_\varepsilon(1)}_{=0} = a_\varepsilon(0)u_\varepsilon'(0)\int_0^1 \frac{1}{a_\varepsilon(z)}\,dz - \int_0^1 \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\,dz \iff$$

$$a_\varepsilon(0)u_\varepsilon'(0) = \left(\int_0^1 \frac{1}{a_\varepsilon(z)}\,dz\right)^{-1}\int_0^1 \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\,dz.$$

Substituting this into (1) gives

$$\boxed{u_\varepsilon(x) = \left(\int_0^1 \frac{1}{a_\varepsilon(z)}\,dz\right)^{-1}\int_0^1 \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\,dz\int_0^x \frac{1}{a_\varepsilon(z)}\,dz - \int_0^x \frac{1}{a_\varepsilon(z)}\int_0^z f(y)\,dy\,dz}$$

as required.

(ii) Taking $\varepsilon = \varepsilon_n = \frac{1}{n}$ gives

$$u_{\varepsilon_n}(x) = \left(\int_0^1 \frac{1}{a(nz)}\,dz\right)^{-1}\int_0^1 \frac{1}{a(nz)}\int_0^z f(y)\,dy\,dz\int_0^x \frac{1}{a(nz)}\,dz - \int_0^x \frac{1}{a(nz)}\int_0^z f(y)\,dy\,dz.$$

We are told in the hint to use the Riemann-Lebesgue Lemma, which states that if $g \in L^\infty(\mathbb{R})$ is 1–periodic, then for any interval $[c, d] \subseteq \mathbb{R}$,

$$\lim_{n\to\infty}\int_c^d g(nz)h(z)\,dz = \int_c^d \overline{g}\,h(z)\,dz \qquad \forall\, h \in L^1(\mathbb{R}) \cap C^1(\mathbb{R}) \tag{2}$$

Applying (2) with $c = 0$, $d = 1$, $g(z) = 1/a(z)$, and $h(z) = 1$ on $[c, d]$ gives

$$\lim_{n\to\infty}\int_0^1 \frac{1}{a(nz)}\,dz = \int_0^1 \overline{\left(\frac{1}{a}\right)}\,dz = \overline{\left(\frac{1}{a}\right)}.$$

(Technical remark: We cannot take $h(z) = 1$ for all $z \in \mathbb{R}$, else $h \notin L^1(\mathbb{R})$. But we can take h to be any function in $L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ such that $h = 1$ on $[c, d]$. The choice of h outside $[c, d]$ does not matter since it does not affect the integrals in (2).)

Applying (2) with $c = 0$, $d = x$, $g(z) = 1/a(z)$ (since a is periodic and bounded below by a positive constant, g is periodic and bounded), and $h(z) = 1$ on $[c, d]$ gives

$$\lim_{n\to\infty}\int_0^x \frac{1}{a(nz)}\,dz = \int_0^x \overline{\left(\frac{1}{a}\right)}\,dz = x\,\overline{\left(\frac{1}{a}\right)}.$$

Applying (2) with $c = 0$, $d = 1$, $g(z) = 1/a(z)$, and $h(z) = \int_0^z f(y)\,dy$ on $[c, d]$ gives

$$\lim_{n\to\infty}\int_0^1 \frac{1}{a(nz)}\int_0^z f(y)\,dy\,dz = \overline{\left(\frac{1}{a}\right)}\int_0^1\int_0^z f(y)\,dy\,dz.$$

Finally, applying (2) with $c = 0$, $d = x$, $g(z) = 1/a(z)$, and $h(z) = \int_0^z f(y)\,dy$ on $[c,d]$ gives

$$\lim_{n\to\infty} \int_0^x \frac{1}{a(nz)} \int_0^z f(y)\,dy\,dz = \overline{\left(\frac{1}{a}\right)} \int_0^x \int_0^z f(y)\,dy\,dz.$$

$$\boxed{\lim_{n\to\infty} u_{\varepsilon_n}(x) = u_0(x) := x\overline{\left(\frac{1}{a}\right)} \int_0^1 \int_0^z f(y)\,dy\,dz - \overline{\left(\frac{1}{a}\right)} \int_0^x \int_0^z f(y)\,dy\,dz.} \tag{3}$$

(iii) This is simply a matter of interchanging the order of integration:

$$\begin{aligned}
u_0(x) &= x\overline{\left(\frac{1}{a}\right)} \int_0^1 \int_0^z f(y)\,dy\,dz - \overline{\left(\frac{1}{a}\right)} \int_0^x \int_0^z f(y)\,dy\,dz \\
&= x\overline{\left(\frac{1}{a}\right)} \int_0^1 \int_y^1 f(y)\,dz\,dy - \overline{\left(\frac{1}{a}\right)} \int_0^x \int_y^x f(y)\,dz\,dy \\
&= x\overline{\left(\frac{1}{a}\right)} \int_0^1 (1-y)f(y)dy - \overline{\left(\frac{1}{a}\right)} \int_0^x (x-y)f(y)\,dy \\
&= \overline{\left(\frac{1}{a}\right)} \left\{ \int_0^x [x(1-y) - (x-y)]f(y)\,dy + \int_x^1 x(1-y)f(y)\,dy \right\} \\
&= \overline{\left(\frac{1}{a}\right)} \left\{ \int_0^x y(1-x)f(y)\,dy + \int_x^1 x(1-y)f(y)\,dy \right\} \\
&= \int_0^1 G(x,y)f(y)\,dy
\end{aligned}$$

with

$$\boxed{G(x,y) = \begin{cases} \overline{\left(\frac{1}{a}\right)} y(1-x) & \text{if } y \le x, \\ \overline{\left(\frac{1}{a}\right)} x(1-y) & \text{if } y \ge x. \end{cases}}$$

(iv) Clearly u_0 satisfies the boundary conditions. By the Fundamental Theorem of Calculus, differentiating equation (3) gives

$$u_0'(x) = \overline{\left(\frac{1}{a}\right)} \int_0^1 \int_0^z f(y)\,dy\,dz - \overline{\left(\frac{1}{a}\right)} \int_0^x f(y)\,dy.$$

Differentiating again gives

$$u_0''(x) = -\overline{\left(\frac{1}{a}\right)} f(x).$$

Therefore

$$-a_0 u_0''(x) = -\frac{1}{\overline{\left(\frac{1}{a}\right)}} \left[-\overline{\left(\frac{1}{a}\right)} f(x) \right] = f(x)$$

as required.

(v) By definition,

$$\overline{a} = \int_0^1 a(x)\,dx = \int_0^{\frac{1}{2}} \frac{1}{2}\,dx + \int_{\frac{1}{2}}^1 1\,dx = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \boxed{\frac{3}{4}}$$

On the other hand,

$$a_0 = \left(\int_0^1 \frac{1}{a(x)}\,dx\right)^{-1} = \left(\int_0^{\frac{1}{2}} 2\,dx + \int_{\frac{1}{2}}^1 1\,dx\right)^{-1} = \left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 1\right)^{-1} = \left(\frac{3}{2}\right)^{-1} = \boxed{\frac{2}{3}}$$

Therefore $a_0 \neq \overline{a}$. In fact the Cauchy-Schwarz inequality can be used to show that

$$a_0 \leq \overline{a}$$

for any choice of a.

(vi) Without loss of generality we can assume that $c > 0$. Using the hint and integration by parts gives

$$\begin{aligned}\int_c^d g(nz)h(z)\,dz &= \int_c^d \left(\frac{1}{n}\int_0^{nz} g(y)\,dy\right)' h(z)\,dz \\ &= \frac{1}{n}\int_0^{nz} g(y)\,dy\,h(z)\bigg|_c^d - \int_c^d \frac{1}{n}\int_0^{nz} g(y)\,dy\,h'(z)\,dz. \end{aligned} \tag{4}$$

Let $z \in [c,d], n \in \mathbb{N}$ and let $\lfloor nz \rfloor \in (nz-1, nz]$ denote floor(nz), which is the largest integer less than or equal to nz. Since a is 1–periodic,

$$\int_0^{nz} g(y)\,dy = \int_0^{\lfloor nz \rfloor} g(y)\,dy + \int_{\lfloor nz \rfloor}^{nz} g(y)\,dy = \lfloor nz \rfloor \int_0^1 g(y)\,dy + \int_{\lfloor nz \rfloor}^{nz} g(y)\,dy. \tag{5}$$

Observe that

$$z - \frac{1}{n} = \frac{nz-1}{n} < \frac{\lfloor nz \rfloor}{n} \leq \frac{nz}{n} = z.$$

Therefore by the Pinching Lemma (Squeezing Lemma)

$$\lim_{n\to\infty} \frac{\lfloor nz \rfloor}{n} = z. \tag{6}$$

Also

$$\left|\frac{1}{n}\int_{\lfloor nz \rfloor}^{nz} g(y)\,dy\right| \leq \frac{1}{n}(nz - \lfloor nz \rfloor)\|g\|_{L^\infty(\mathbb{R})} \leq \frac{1}{n}\,\|g\|_{L^\infty(\mathbb{R})}.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n}\int_{\lfloor nz \rfloor}^{nz} g(y)\,dy = 0. \tag{7}$$

By combining equations (5), (6), (7) we find that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^{nz} g(y)\,dy = z\int_0^1 g(y)\,dy. \tag{8}$$

Therefore the limit of the first term on the right-hand side of equation (4) is

$$\lim_{n\to\infty} \frac{1}{n}\int_0^{nz} g(y)\,dy\,h(z)\bigg|_c^d = z\int_0^1 g(y)\,dy\,h(z)\bigg|_c^d. \tag{9}$$

Now we find the limit of the second term on the right-hand side of (4). By the computations above

$$\begin{aligned}
&\left|\int_c^d \frac{1}{n}\int_0^{nz} g(y)\,dy\,h'(z)\,dz - \int_c^d z\int_0^1 g(y)\,dy\,h'(z)\,dz\right| \\
&\le \int_c^d \left|\frac{1}{n}\int_0^{nz} g(y)\,dy - z\int_0^1 g(y)\,dy\right| |h'(z)|\,dz \\
&\le \int_c^d \left|\frac{\lfloor nz\rfloor}{n}\int_0^1 g(y)\,dy + \frac{1}{n}\int_{\lfloor nz\rfloor}^{nz} g(y)\,dy - z\int_0^1 g(y)\,dy\right| |h'(z)|\,dz \\
&\le \int_c^d \left(\left|\frac{\lfloor nz\rfloor}{n}\int_0^1 g(y)\,dy - z\int_0^1 g(y)\,dy\right| + \frac{1}{n}\int_{\lfloor nz\rfloor}^{nz} |g(y)|\,dy\right) |h'(z)|\,dz \\
&\le \int_c^d \left|\frac{\lfloor nz\rfloor - nz}{n}\int_0^1 g(y)\,dy\right| |h'(z)|\,dz + \frac{1}{n}\|g\|_{L^\infty(\mathbb{R})}\|h'\|_{L^1([c,d])} \\
&\le \int_c^d \frac{1}{n}\int_0^1 |g(y)|\,dy\,|h'(z)|\,dz + \frac{1}{n}\|g\|_{L^\infty(\mathbb{R})}\|h'\|_{L^1([c,d])} \\
&\le \frac{2}{n}\|g\|_{L^\infty(\mathbb{R})}\|h'\|_{L^1([c,d])} \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\int_c^d \frac{1}{n}\int_0^{nz} g(y)\,dy\,h'(z)\,dz = \int_c^d z\int_0^1 g(y)\,dy\,h'(z)\,dz. \tag{10}$$

Combining (4), (9), (10) and then integrating by parts yields

$$\begin{aligned}
\lim_{n\to\infty}\int_c^d g(nz)h(z)\,dz &= z\int_0^1 g(y)\,dy\,h(z)\Big|_c^d - \int_c^d z\int_0^1 g(y)\,dy\,h'(z)\,dz \\
&= \int_c^d\int_0^1 g(y)\,dy\,h(z)\,dz \\
&= \int_c^d \overline{g}\,h(z)\,dz
\end{aligned}$$

as required.

3. *Radial symmetry of Laplace's equation on* $\mathbb{R}^n$. Let $v:\mathbb{R}^n\to\mathbb{R}$ be a harmonic function. Let $R\in O(n,\mathbb{R})$ and define $w:\mathbb{R}^n\to\mathbb{R}$ by $w(\boldsymbol{x}) := v(R\boldsymbol{x})$. Then

$$\begin{aligned}
w_{x_i} &= \sum_{j=1}^n \frac{\partial v}{\partial x_j}\frac{\partial (R\boldsymbol{x})_j}{\partial x_i} = \sum_{j=1}^n \frac{\partial v}{\partial x_j}\frac{\partial}{\partial x_i}\sum_{k=1}^n R_{jk}x_k \\
&= \sum_{j=1}^n \frac{\partial v}{\partial x_j}\sum_{k=1}^n R_{jk}\frac{\partial x_k}{\partial x_i} = \sum_{j=1}^n \frac{\partial v}{\partial x_j}\sum_{k=1}^n R_{jk}\delta_{ki} \\
&= \sum_{j=1}^n \frac{\partial v}{\partial x_j}R_{ji}.
\end{aligned}$$

To be precise

$$w_{x_i}(\boldsymbol{x}) = \sum_{j=1}^n v_{x_j}(R\boldsymbol{x})R_{ji}.$$

(This can also be written as $\nabla w(\boldsymbol{x}) = R^T\nabla v(R\boldsymbol{x})$.)

Now we compute the second partial derivatives:

$$
\begin{aligned}
w_{x_i x_i} &= \frac{\partial}{\partial x_i} \sum_{j=1}^{n} v_{x_j}(R\boldsymbol{x}) R_{ji} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial v_{x_j}}{\partial x_k} \frac{\partial (R\boldsymbol{x})_k}{\partial x_i} R_{ji} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} R_{ji} \frac{\partial}{\partial x_i} \sum_{l=1}^{n} R_{kl} x_l \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} R_{ji} \sum_{l=1}^{n} R_{kl} \frac{\partial x_l}{\partial x_i} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} R_{ji} \sum_{l=1}^{n} R_{kl} \delta_{il} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} R_{ji} R_{ki}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Delta w &= \sum_{i=1}^{n} w_{x_i x_i} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} R_{ji} R_{ki} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} \sum_{i=1}^{n} R_{ji} R_{ki} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} \sum_{i=1}^{n} R_{ji} (R^T)_{ik} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} (RR^T)_{jk} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} I_{jk} \qquad (11)
\end{aligned}
$$

since R is an orthogonal matrix. There are two ways to conclude from here: If are are familiar with the matrix inner product, then (11) gives

$$\Delta w = D^2 v : I = \text{trace}(D^2 v) = \Delta v = 0$$

since v is harmonc. Otherwise we can continue from (11) using indices:

$$\Delta w = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} I_{jk} = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{x_j x_k} \delta_{jk} = \sum_{j=1}^{n} v_{x_j x_j} = \Delta v = 0,$$

as required.

4. *Fundamental solution of Poisson's equation in 3D.*

(i) One way of computing $\|\Phi\|_{L^1(B_R(\mathbf{0}))}$ is using spherical polar coordinates:

$$\begin{aligned}
\|\Phi\|_{L^1(B_R(\mathbf{0}))} &= \int_{B_R(\mathbf{0})} |\Phi(\boldsymbol{x})|\, d\boldsymbol{x} \\
&= \frac{1}{4\pi} \int_{B_R(\mathbf{0})} \frac{1}{|\boldsymbol{x}|}\, d\boldsymbol{x} \\
&= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} \frac{1}{r} r^2 \sin\theta\, dr d\theta d\phi \\
&= \frac{1}{4\pi} \int_0^{2\pi} 1\, d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^R r\, dr \\
&= \boxed{\frac{R^2}{2}}
\end{aligned}$$

Another way of computing $\|\Phi\|_{L^1(B_R(\mathbf{0}))}$ is as follows:

$$\begin{aligned}
\|\Phi\|_{L^1(B_R(\mathbf{0}))} &= \int_{B_R(\mathbf{0})} |\Phi(\boldsymbol{x})|\, d\boldsymbol{x} \\
&= \int_0^R \left(\int_{\partial B_r(\mathbf{0})} |\Phi(\boldsymbol{y})|\, dS(\boldsymbol{y}) \right) dr \\
&= \int_0^R \left(\int_{\partial B_r(\mathbf{0})} \frac{1}{4\pi} \frac{1}{|\boldsymbol{y}|}\, dS(\boldsymbol{y}) \right) dr \\
&= \frac{1}{4\pi} \int_0^R \left(\int_{\partial B_r(\mathbf{0})} \frac{1}{r}\, dS(\boldsymbol{y}) \right) dr \\
&= \frac{1}{4\pi} \int_0^R \left(\operatorname{area}(\partial B_r(\mathbf{0}))\, \frac{1}{r} \right) dr \\
&= \frac{1}{4\pi} \int_0^R \left(4\pi r^2 \frac{1}{r} \right) dr \\
&= \int_0^R r\, dr \\
&= \boxed{\frac{R^2}{2}}
\end{aligned}$$

(ii) Let $K \subset \mathbb{R}^3$ be compact. Since K is bounded, there exists $R > 0$ such that $K \subset B_R(\mathbf{0})$. Therefore

$$\int_K |\Phi(\boldsymbol{x})|\, d\boldsymbol{x} \le \int_{B_R(\mathbf{0})} |\Phi(\boldsymbol{x})|\, d\boldsymbol{x} = \frac{R^2}{2} < \infty.$$

Therefore $\Phi \in L^1_{\text{loc}}(\mathbb{R}^3)$.

(iii) By part (i),

$$\lim_{R\to\infty} \|\Phi\|_{L^1(B_R(\mathbf{0}))} = \lim_{R\to\infty} \frac{R^2}{2} = +\infty.$$

Therefore $\Phi \notin L^1(\mathbb{R}^3)$.

(iv) By the Chain Rule

$$\nabla\Phi(\boldsymbol{x}) = \frac{1}{4\pi}\left(-\frac{1}{|\boldsymbol{x}|^2}\right)\nabla|\boldsymbol{x}| = \frac{1}{4\pi}\left(-\frac{1}{|\boldsymbol{x}|^2}\right)\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = -\frac{1}{4\pi}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}.$$

Let $K \subset \mathbb{R}^3$ be compact. Since K is bounded, there exists $R > 0$ such that $K \subset B_R(\mathbf{0})$. Therefore

$$\begin{aligned}
\int_K |\nabla\Phi(\boldsymbol{x})|\, d\boldsymbol{x} &\le \int_{B_R(\mathbf{0})} |\nabla\Phi(\boldsymbol{x})|\, d\boldsymbol{x} \\
&= \int_{B_R(\mathbf{0})} \frac{1}{4\pi}\frac{1}{|\boldsymbol{x}|^2}\, d\boldsymbol{x} \\
&= \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} \frac{1}{r^2} r^2 \sin\theta\, dr d\theta d\phi \\
&= \frac{1}{4\pi}\int_0^{2\pi} 1\, d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^R 1\, dr \\
&= R \\
&< \infty.
\end{aligned}$$

Therefore $\nabla\Phi \in L^1_{\text{loc}}(\mathbb{R}^3)$.

5. *Fundamental solution of Poisson's equation in 1D.* We compute

$$\begin{aligned}
u''(x) &= (\Phi * f)''(x) \\
&= (f * \Phi)''(x) && \text{(symmetry of convolution)} \\
&= \frac{d^2}{dx^2}\int_{-\infty}^{\infty} \Phi(y) f(x-y)\, dy \\
&= \int_{-\infty}^{\infty} \Phi(y)\frac{d^2}{dx^2} f(x-y)\, dy \\
&= \int_{-\infty}^{0} y\,\frac{d^2}{dx^2} f(x-y)\, dy \\
&= \int_{-\infty}^{0} y\,\frac{d^2}{dy^2} f(x-y)\, dy \\
&= y\,\frac{d}{dy} f(x-y)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{d}{dy} f(x-y)\, dy && \text{(integration by parts)} \\
&= -f(x) && \text{(Fundamental Theorem of Calculus)}
\end{aligned}$$

as required.

6. *The function spaces L^1 and L^1_{loc}.* Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^k$, $k \in \mathbb{R}$. By integrating we see that

(i) $f \in L^1((-R,R))$ for $k > -1$,

(ii) $f \in L^1((R,\infty))$ for $k < -1$,

(iii) $f \in L^1_{\text{loc}}(\mathbb{R})$ for $k > -1$,

(iv) $f \notin L^1(\mathbb{R})$ for any k (by parts (i),(ii)).

7. *Properties of the convolution.*

(i) Let $\varphi \in L^1_{\text{loc}}(\mathbb{R})$, $f \in C_c(\mathbb{R})$ and let $K = \text{supp}(f)$. Choose $R > 0$ such that $K \subset [-R,R]$. In particular, $f = 0$ outside the interval $[-R,R]$. Therefore

$$\begin{aligned}
|(\varphi * f)(x)| &= \left| \int_{-\infty}^{\infty} \varphi(x-y) f(y)\, dy \right| \\
&= \left| \int_{-R}^{R} \varphi(x-y) f(y)\, dy \right| \\
&\le \int_{-R}^{R} |\varphi(x-y)||f(y)|\, dy \\
&\le \max_{[-R,R]} |f| \int_{-R}^{R} |\varphi(x-y)|\, dy \\
&= \max_{[-R,R]} |f| \int_{-R-x}^{R-x} |\varphi(z)|\, dz \\
&< \infty
\end{aligned}$$

since $\varphi \in L^1_{\text{loc}}(\mathbb{R})$ and $[-R-x, R-x]$ is compact.

(ii) Now assume that $\varphi \in L^1(\mathbb{R})$. By Lemma 4.12, $f \in L^\infty(\mathbb{R})$. Therefore

$$\begin{aligned}
|(\varphi * f)(x)| &\le \int_{-\infty}^{\infty} |\varphi(x-y)||f(y)|\, dy \\
&\le \sup_{y \in \mathbb{R}} |f(y)| \int_{-\infty}^{\infty} |\varphi(x-y)|\, dy \\
&= \sup_{y \in \mathbb{R}} |f(y)| \int_{-\infty}^{\infty} |\varphi(z)|\, dz \\
&= \|f\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Therefore

$$\|\varphi * f\|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |(\varphi * f)(x)| \le \|f\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^1(\mathbb{R})} < \infty$$

and so $\varphi * f \in L^\infty(\mathbb{R})$, as required.

(iii) The convolution is commutative since

$$\begin{aligned}
(\varphi * f)(x) &= \int_{-\infty}^{\infty} \varphi(x-y) f(y)\, dy \\
&= \int_{\infty}^{-\infty} \varphi(z) f(x-z)(-1) dz && (z = x - y) \\
&= \int_{-\infty}^{\infty} \varphi(z) f(x-z) dz \\
&= (f * \varphi)(x)
\end{aligned}$$

as required.

8. *The Poincaré inequality for functions that vanish on the boundary.* Let $f \in C^1([a,b])$ satisfy $f(a) = f(b) = 0$. Then

$$f(x) = f(a) + \int_a^x f'(y)\, dy = \int_a^x f'(y)\, dy$$

since $f(a) = 0$. Therefore

$$
\begin{aligned}
|f(x)| &= \left| \int_a^x f'(y)\,dy \right| \\
&= \left| \int_a^x 1 \cdot f'(y)\,dy \right| \\
&\le \left| \int_a^x 1^2\,dy \right|^{1/2} \left| \int_a^x |f'(y)|^2\,dy \right|^{1/2} && \text{(Cauchy-Schwarz)} \\
&\le (x-a)^{1/2} \left(\int_a^b |f'(y)|^2\,dy \right)^{1/2}.
\end{aligned}
$$

Squaring and integrating gives

$$
\begin{aligned}
\int_a^b |f(x)|^2\,dx &\le \int_a^b (x-a) \int_a^b |f'(y)|^2\,dy\,dx \\
&= \int_a^b (x-a)\,dx \int_a^b |f'(y)|^2\,dy \\
&= \frac{1}{2}(x-a)^2\Big|_a^b \int_a^b |f'(y)|^2\,dy \\
&= \frac{1}{2}(b-a)^2 \int_a^b |f'(y)|^2\,dy.
\end{aligned}
$$

This is the Poincaré inequality with $C = \frac{1}{2}(b-a)^2$.

9. *The Poincaré inequality on unbounded domains.*

   (i) For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$
f_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, -n-1], \\ (x-(-n-1))^2(x-(-n+1))^2 & \text{if } x \in [-n-1, -n], \\ 1 & \text{if } x \in [-n, n], \\ (x-(n+1))^2(x-(n-1))^2 & \text{if } x \in [n, n+1], \\ 0 & \text{if } x \in [n+1, \infty). \end{cases}
$$

   (Exercise: Sketch f_n to get a better understanding of the example.) Observe that

$$
\begin{gathered}
f_n(-n-1) = f_n(n+1) = 0, \\
f_n(-n) = f_n(n) = 1, \\
f_n'(-n-1) = f_n'(-n) = f_n'(n) = f_n'(n+1) = 0.
\end{gathered}
$$

   Therefore $f_n \in C^1(\mathbb{R})$. We also have $f_n \in L^2(\mathbb{R})$ since

$$
\|f_n\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |f_n(x)|^2\,dx < \int_{-n-1}^{n+1} 1\,dx = 2(n+1).
$$

   We compute

$$
\begin{aligned}
\|f_n'\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} |f_n'(x)|^2\, dx \\
&= 2\int_n^{n+1} \left[\frac{d}{dx}(x-(n+1))^2(x-(n-1))^2\right]^2\, dx \\
&= 2\int_n^{n+1} \left[2(x-(n+1))(x-(n-1))^2 + 2(x-(n+1))^2(x-(n-1))\right]^2\, dx \\
&= 2\int_0^1 \left[2(y-1)(y+1)^2 + 2(y-1)^2(y+1)\right]^2\, dy && (y = x-n)
\end{aligned}
$$

which is independent of n. But

$$
\|f_n\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |f_n(x)|^2\, dx > \int_{-n}^{n} |f_n(x)|^2\, dx = 2n.
$$

Therefore

$$
\|f_n'\|_{L^2(\mathbb{R})} = \text{constant}, \qquad \|f_n\|_{L^2(\mathbb{R})} \overset{n\to\infty}{\longrightarrow} \infty
$$

as required. This means that, given any $C > 0$, we can choose N large enough so that

$$
\int_{-\infty}^{\infty} |f_N(x)|^2\, dx > C\int_{-\infty}^{\infty} |f_N'(x)|^2\, dx,
$$

which means that the Poincaré inequality on $\mathbb{R}$ does not hold. We constructed this counter example using *spreading*; the support of f_n spreads as $n \to \infty$ without changing the L^2–norm of f_n'.

(ii) Let $\Omega = (a,b) \times (-\infty, \infty)$. Let $f \in C^1(\overline{\Omega}) \cap L^2(\Omega)$ with $\nabla f \in L^2(\Omega)$ and with $f(a,y) = f(b,y) = 0$ for all $y \in \mathbb{R}$. Then

$$
\begin{aligned}
\int_\Omega |f(\boldsymbol{x})|^2\, d\boldsymbol{x} &= \int_{-\infty}^{\infty}\left(\int_a^b |f(x,y)|^2\, dx\right) dy \\
&\le \int_{-\infty}^{\infty}\left(C\int_a^b |f_x(x,y)|^2\, dx\right) dy && \text{(Poincaré inequality in } x\text{)} \\
&\le C\int_{-\infty}^{\infty}\int_a^b (|f_x(x,y)|^2 + |f_y(x,y)|^2)\, dxdy \\
&= C\int_\Omega |\nabla f(\boldsymbol{x})|^2\, d\boldsymbol{x}
\end{aligned}
$$

as required.

10. *The Poincaré constant depends on the domain.* There exits $C_1 > 0$ such that

$$
\int_0^1 |f(x)|^2\, dx \le C_1 \int_0^1 |f'(x)|^2\, dx \tag{12}
$$

for all $f \in C^1([0,1])$ with $f(0) = f(1) = 0$. Let $g \in C^1([0,L])$ with $g(0) = g(L) = 0$. Then

$$\begin{aligned}
\int_0^L |g(x)|^2\,dx &= \int_0^1 |g(Ly)|^2 L\,dy && (y = x/L)\\
&= L\int_0^1 |f(y)|^2\,dy && (f(y) := g(Ly))\\
&\le LC_1\int_0^1 |f'(y)|^2\,dy && (\text{equation (12)})\\
&= LC_1\int_0^1 |Lg'(Ly)|^2\,dy && (f'(y) = Lg'(Ly))\\
&= L^3C_1\int_0^1 |g'(Ly)|^2\,dy\\
&= L^2C_1\int_0^L |g'(x)|^2\,dx && (y = x/L)\\
&= C_L\int_0^L |g'(x)|^2\,dx
\end{aligned}$$

with $C_L = L^2C_1$, as desired.

11. *Eigenvalues of $-\Delta$: Can you hear the shape of a drum?* Multiply the PDE $-\Delta u = \lambda u$ by $\overline{u}$ (the complex conjugate of u) and integrate over Ω:

$$-\int_\Omega \overline{u}\,\Delta u\,d\boldsymbol{x} = \lambda\int_\Omega \overline{u}u\,d\boldsymbol{x} \quad\Longleftrightarrow\quad -\int_{\partial\Omega}\overline{u}\,\nabla u\cdot\boldsymbol{n}\,dL + \int_\Omega \nabla\overline{u}\cdot\nabla u\,d\boldsymbol{x} = \lambda\int_\Omega |u|^2\,d\boldsymbol{x}.$$

The boundary condition $u = 0$ on $\partial\Omega$ implies that $\overline{u} = 0$ on $\partial\Omega$ and so

$$\begin{aligned}
\int_\Omega \overline{\nabla u}\cdot\nabla u\,d\boldsymbol{x} = \lambda\int_\Omega |u|^2\,d\boldsymbol{x} \quad&\Longleftrightarrow\quad \int_\Omega |\nabla u|^2\,d\boldsymbol{x} = \lambda\int_\Omega |u|^2\,d\boldsymbol{x}\\
&\Longleftrightarrow\quad \lambda = \frac{\int_\Omega |\nabla u|^2\,d\boldsymbol{x}}{\int_\Omega |u|^2\,d\boldsymbol{x}} > 0
\end{aligned}$$

as required.

12. *The optimal Poincaré constant and eigenvalues of $-\Delta$.*

   (i) Multiply the PDE $-\Delta u = \lambda u$ by u and integrate over Ω:

$$-\int_\Omega u\Delta u\,d\boldsymbol{x} = \lambda\int_\Omega u^2\,d\boldsymbol{x} \quad\Longleftrightarrow\quad \int_\Omega |\nabla u|^2\,d\boldsymbol{x} = \lambda\int_\Omega u^2\,d\boldsymbol{x}$$

   since $u = 0$ on $\partial\Omega$. Rearranging gives

$$\lambda = \frac{\int_\Omega |\nabla u|^2\,d\boldsymbol{x}}{\int_\Omega |u|^2\,d\boldsymbol{x}}.$$

   (ii) Let $u \in C^2(\overline{\Omega}) \cap V$ minimise E. Let $\varphi \in V$. Define $u_\varepsilon = u + \varepsilon\varphi \in V$ and define $g(\varepsilon) = E[u_\varepsilon]$, $\varepsilon \in \mathbb{R}$.

Since E is minimised by u, then g is minimised by 0. It follows that

$$\begin{aligned}
0 &= g'(0) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E[u_\varepsilon] \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{\displaystyle\int_\Omega |\nabla u_\varepsilon|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u_\varepsilon|^2 \, d\boldsymbol{x}} \\
&= \frac{2\displaystyle\int_\Omega \nabla u \cdot \nabla \varphi \, d\boldsymbol{x} \int_\Omega |u|^2 \, d\boldsymbol{x} - 2\int_\Omega |\nabla u|^2 \, d\boldsymbol{x} \int_\Omega u\varphi \, d\boldsymbol{x}}{\left(\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}\right)^2}.
\end{aligned}$$

The numerator must be zero. Rearranging gives

$$\int_\Omega \nabla u \cdot \nabla \varphi \, d\boldsymbol{x} = \left(\frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}} \right) \int_\Omega u\varphi \, d\boldsymbol{x}.$$

Integrating by parts gives

$$-\int_\Omega \Delta u \, \varphi \, d\boldsymbol{x} = \left(\frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}} \right) \int_\Omega u\varphi \, d\boldsymbol{x}.$$

Since this holds for all $\varphi \in V$, the Fundamental Lemma of the Calculus of Variations implies that

$$-\Delta u = \left(\frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}} \right) u \quad \text{in } \Omega.$$

If we define

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}},$$

then

$$-\Delta u = \lambda u \quad \text{in } \Omega.$$

In other words, u is an eigenfunction of $-\Delta$. By definition

$$E[u] = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}}{\displaystyle\int_\Omega |u|^2 \, d\boldsymbol{x}} = \lambda.$$

Since u minimises E, then λ must be the smallest eigenvalue of $-\Delta$ on V, i.e., $\lambda = \lambda_1$, otherwise we obtain a contradiction. Therefore $E[u] = \lambda_1$, as required.

(iii) Let $C > 0$ satisfy

$$\|f\|_{L^2(\Omega)} \le C\|\nabla f\|_{L^2(\Omega)}$$

for all $f \in C^1(\overline{\Omega})$ with $f = 0$ on $\partial\Omega$. Then

$$\frac{1}{C} \le \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}$$

for all $f \in V$ and so

$$\frac{1}{C} \le \inf_{f\in V} \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}.$$

The smallest value of C satisfying this inequality is $C = C_{\mathrm{P}}$ where

$$\frac{1}{C_{\mathrm{P}}} = \inf_{f\in V} \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}.$$

(iv) Combining parts (ii) and (iii) gives

$$\frac{1}{C_{\mathrm{P}}} = \inf_{f\in V} \frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} = \inf_{f\in V} E[v]^{1/2} = \left(\inf_{f\in V} E[v]\right)^{1/2} = \sqrt{\lambda_1}.$$

Therefore

$$C_{\mathrm{P}} = \frac{1}{\sqrt{\lambda_1}}$$

as desired.

(v) If $\Omega = (0, 2\pi)$, then the corresponding eigenvalue problem is

$$-u'' = \lambda u \quad \text{in } (0, 2\pi), \qquad u(0) = u(2\pi) = 0.$$

The eigenfunctions are $u_n(x) = \sin\left(\frac{nx}{2}\right)$ (see Exercise Sheet 5, Q16) and the corresponding eigenvalues are $\lambda_n = n^2/4$, $n \in \mathbb{N}$. Therefore $\lambda_1 = 1/4$ and $C_{\mathrm{P}} = 1/\sqrt{1/4} = 2$. In Q8 we obtained the Poincaré constant $(b-a)/\sqrt{2} = \sqrt{2}\pi$, which is obviously much bigger than the optimal constant $C_{\mathrm{P}} = 2$.

13. *Uniqueness for Poisson's equation with Robin boundary conditions.* Let u_1 and u_2 be solutions of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} + \alpha u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Let $w = u_1 - u_2$. Since the PDE is linear, subtracting the equations satisfied by u_1 and u_2 gives

$$\begin{aligned} -\Delta w &= 0 \quad \text{in } \Omega, \\ \nabla w \cdot \boldsymbol{n} + \alpha w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Multiply $-\Delta w = 0$ by w and integrate by parts over Ω:

$$\begin{aligned} -\int_\Omega w\Delta w \, d\boldsymbol{x} = 0 \quad &\iff \quad -\int_{\partial\Omega} w \, \nabla w \cdot \boldsymbol{n} \, dS + \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} = 0 \\ &\iff \quad \alpha \int_{\partial\Omega} w^2 \, dS + \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} = 0 \end{aligned}$$

since $\nabla w \cdot \boldsymbol{n} = -\alpha w$ on $\partial\Omega$. But $\alpha > 0$. Therefore

$$\int_{\partial\Omega} w^2\, dS = 0, \qquad \int_\Omega |\nabla w|^2\, d\boldsymbol{x} = 0.$$

The second equation implies that $\nabla w = \mathbf{0}$ and hence $w =$ constant (or at least constant on each connected component of Ω). The first equation implies that this constant must be zero. Therefore $w = 0$ and $u_1 = u_2$, as required.

14. *Uniqueness for a more general elliptic problem.* Consider the linear, second-order, elliptic PDE

$$\begin{aligned} -\operatorname{div}(A\,\nabla u) + \boldsymbol{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

(i) Suppose that $u_1, u_2 \in C^2(\overline{\Omega})$ satisfy (13). Let $w = u_1 - u_2$. Since the PDE is linear, subtracting the equations satisfied by u_1 and u_2 gives

$$\begin{aligned} -\operatorname{div}(A\,\nabla w) + \boldsymbol{b}\cdot\nabla w + cw &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

Clearly $w = 0$ satisfies (14). We want to show that it is the only solution. Multiply the PDE for w by w and integrate over Ω:

$$\begin{aligned} 0 &= \int_\Omega w(-\operatorname{div}(A\,\nabla w) + \boldsymbol{b}\cdot\nabla w + cw)\, d\boldsymbol{x} \\ &= -\int_\Omega w\operatorname{div}(A\,\nabla w)\, d\boldsymbol{x} + \int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} + \int_\Omega cw^2\, d\boldsymbol{x} \\ &= -\int_{\partial\Omega} w(A\,\nabla w)\cdot\boldsymbol{n}\, dS + \int_\Omega \nabla w\cdot(A\,\nabla w)\, d\boldsymbol{x} + \int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} + \int_\Omega cw^2\, d\boldsymbol{x} \\ &= \int_\Omega \nabla w\cdot(A\,\nabla w)\, d\boldsymbol{x} + \int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} + \int_\Omega cw^2\, d\boldsymbol{x} \end{aligned} \tag{15}$$

since $w = 0$ on $\partial\Omega$. Observe that

$$\int_\Omega \nabla w\cdot(A\,\nabla w)\, d\boldsymbol{x} = \int_\Omega (\nabla w)^{\mathrm{T}} A\,\nabla w\, d\boldsymbol{x} \geq \alpha \int_\Omega |\nabla w|^2\, d\boldsymbol{x} \tag{16}$$

by the assumption that A is uniformly positive definite (take $\boldsymbol{y} = \nabla w$ in $\boldsymbol{y}^{\mathrm{T}} A(\boldsymbol{x})\boldsymbol{y} \geq \alpha|\boldsymbol{y}|^2$). Integrating by parts gives

$$\begin{aligned} \int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} &= \int_{\partial\Omega} w^2\,\boldsymbol{b}\cdot\boldsymbol{n}\, dS - \int_\Omega w\operatorname{div}(w\boldsymbol{b})\, d\boldsymbol{x} \\ &= -\int_\Omega w\operatorname{div}(w\boldsymbol{b})\, d\boldsymbol{x} && (w = 0 \text{ on } \partial\Omega) \\ &= -\int_\Omega w\,(\nabla w\cdot\boldsymbol{b} + w\operatorname{div}\boldsymbol{b})\, d\boldsymbol{x} && \text{(product rule)} \\ &= -\int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} \end{aligned}$$

by the assumption that $\operatorname{div}\boldsymbol{b} = 0$. Therefore

$$\int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} = -\int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} \quad\Longleftrightarrow\quad \int_\Omega w\,\boldsymbol{b}\cdot\nabla w\, d\boldsymbol{x} = 0. \tag{17}$$

Combining (15), (16), (17) yields

$$\alpha \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} + \int_\Omega c w^2 \, d\boldsymbol{x} \le 0.$$

But $c \ge 0$ by assumption. Therefore

$$\alpha \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} = 0$$

and so $\nabla w = \mathbf{0}$ in Ω. Hence w is constant (or at least constant on each connected component of Ω). But $w = 0$ on $\partial\Omega$. Therefore $w = 0$, as required.

(ii) The idea is the same as for (i). Let u_n be the unique solution to the PDE with A_n and let u be the unique solution to the PDE with the matrix A. Define $w_n := u_n - u$. We need to show that $\nabla w_n \to 0$ in $L^2(\Omega)$ as $n \to +\infty$. Taking the two PDEs, subtracting them and multiplying the resulting PDE by w_n, we obtain

$$0 = \int_\Omega w_n(-\mathrm{div}(A_n \nabla u_n) + \mathrm{div}(A \nabla u) + \boldsymbol{b} \cdot \nabla w_n + c w_n) \, d\boldsymbol{x}. \tag{18}$$

Proceeding exactly as in (i), we find

$$\int_\Omega w_n \boldsymbol{b} \cdot \nabla w_n \, d\boldsymbol{x} = 0.$$

Moreover, we compute (using integration by parts, since $w_n = 0$ on $\partial\Omega$)

$$\begin{aligned}
&\int_\Omega w_n[-\mathrm{div}(A_n \nabla u_n) + \mathrm{div}(A \nabla u)] \, d\boldsymbol{x} \\
&= \int_\Omega w_n[-\mathrm{div}(A_n \nabla u_n) + \mathrm{div}(A_n \nabla u) - \mathrm{div}(A_n \nabla u) + \mathrm{div}(A \nabla u)] \, d\boldsymbol{x} \\
&= \int_\Omega w_n[-\mathrm{div}(A_n (\nabla u_n - \nabla u)) - \mathrm{div}((A_n - A) \nabla u)] \, d\boldsymbol{x} \\
&= \int_\Omega [\nabla w_n \cdot (A_n \nabla w_n) + \nabla w_n \cdot ((A_n - A) \nabla u)] \, d\boldsymbol{x} \\
&\ge \int_\Omega [\alpha |\nabla w_n|^2 + \nabla w_n \cdot ((A_n - A) \nabla u)] \, d\boldsymbol{x}
\end{aligned}$$

So, all these arguments yield

$$\int_\Omega [\alpha |\nabla w_n|^2 + \nabla w_n \cdot ((A_n - A) \nabla u) + c w_n^2] \, d\boldsymbol{x} \le 0.$$

Now, for any $\varepsilon > 0$, Young's inequality yields

$$\int_\Omega \nabla w_n \cdot ((A_n - A) \nabla u) \, d\boldsymbol{x} = -\frac{\varepsilon}{2} \int_\Omega |\nabla w_n|^2 \, d\boldsymbol{x} - \int_\Omega \frac{1}{2\varepsilon} |A_n - A|^2 |\nabla u|^2 \, d\boldsymbol{x}.$$

By setting $\varepsilon := \alpha$, the previous two identities imply

$$\int_\Omega \left[\frac{\alpha}{2} |\nabla w_n|^2 + c w_n^2 \right] \, d\boldsymbol{x} \le \frac{1}{2\alpha} \|A_n - A\|_{L^\infty}^2 \int_\Omega |\nabla u|^2 \, d\boldsymbol{x}.$$

And by the non-negative property of c, one has

$$\int_\Omega \frac{\alpha}{2} |\nabla w_n|^2 \, d\boldsymbol{x} \le \frac{1}{2\alpha} \|A_n - A\|_{L^\infty}^2 \int_\Omega |\nabla u|^2 \, d\boldsymbol{x}.$$

We conclude by the facts that $\int_\Omega |\nabla u|^2 \, d\boldsymbol{x}$ is bounded and $\|A_n - A\|_{L^\infty} \to 0$, as $n \to +\infty$.

15. *Uniqueness for a degenerate diffusion equation.* Clearly $u = \pi$ satisfies

$$\begin{aligned} \Delta u^m &= 0 \quad \text{in } \Omega, \\ u &= \pi \quad \text{on } \partial\Omega. \end{aligned}$$

We use the energy method to show that it is the only positive solution. Let v be any positive solution. Subtracting the PDE for u from the PDE for v and multiplying by $(v - u)$ gives

$$0 = (v - u)(\Delta v^m - \Delta u^m) = (v - \pi)(\Delta v^m - \Delta \pi^m) = (v - \pi)\Delta v^m.$$

Now integrate over Ω:

$$\begin{aligned} 0 &= \int_\Omega (v - \pi)\Delta v^m \, d\boldsymbol{x} \\ &= \int_\Omega (v - \pi) \operatorname{div}\nabla(v^m) \, d\boldsymbol{x} && (\Delta = \operatorname{div}\nabla) \\ &= \int_\Omega (v - \pi) \operatorname{div}(m v^{m-1}\nabla v) \, d\boldsymbol{x} && \text{(Chain Rule)} \\ &= \int_{\partial\Omega} \underbrace{(v - \pi)}_{=0} \, m \, v^{m-1} \nabla v \cdot \boldsymbol{n} \, dS - \int_\Omega \underbrace{\nabla(v - \pi)}_{=\nabla v} \cdot m v^{m-1} \nabla v \, d\boldsymbol{x} && \text{(Integration by parts)} \\ &= -\int_\Omega m v^{m-1} |\nabla v|^2 \, d\boldsymbol{x}. \end{aligned}$$

Therefore

$$\int_\Omega m v^{m-1} |\nabla v|^2 \, d\boldsymbol{x} = 0.$$

But $v > 0$, by assumption. Hence $\nabla v = \boldsymbol{0}$ in Ω and so v is constant in Ω. Since $v = \pi$ on $\partial\Omega$, we conclude that $v = \pi$ everywhere, as required.

16. *The H_0^1 and H^1 norms.*

 (i) We need to check that $\| \cdot \|_{L^2([a,b])}$ satisfies the three properties of a norm: positivity, 1–homogeneity, and the triangle inequality. First we prove positivity. Let $f \in C([a,b])$. Clearly $\|f\|_{L^2([a,b])} \geq 0$. Suppose that $\|f\|_{L^2([a,b])} = 0$ and assume for contradiction that $f \neq 0$. Since f is continuous, then there exists $x_0 \in (a,b)$, $h > 0$ and $\varepsilon > 0$ such that $|f(x)| > \varepsilon$ for all $x \in (x_0 - h, x_0 + h)$. Therefore

$$\|f\|_{L^2([a,b])}^2 \geq \int_{x_0-h}^{x_0+h} |f(x)|^2 \, dx \geq \int_{x_0-h}^{x_0+h} \varepsilon^2 \, dx = 2h\varepsilon^2 > 0,$$

 which is a contradiction. Second we check that $\| \cdot \|_{L^2([a,b])}$ is 1–homogeneous. Let $\lambda \in \mathbb{R}$. Then

$$\|\lambda f\|_{L^2([a,b])} = \left(\int_a^b |\lambda f(x)|^2\right)^{1/2} = |\lambda| \left(\int_a^b |f(x)|^2\right)^{1/2} = |\lambda| \, \|f\|_{L^2([a,b])}$$

as required. Finally, we prove the triangle inequality. Let $f, g \in C([a,b])$. Then

$$\begin{aligned}
\|f+g\|^2_{L^2([a,b])} &= \int_a^b (f(x)+g(x))^2 \, dx \\
&= \int_a^b f(x)^2 \, dx + 2\int_a^b f(x)g(x) \, dx + \int_a^b g(x)^2 \, dx \\
&\le \int_a^b f(x)^2 \, dx + 2\left(\int_a^b f(x)^2 \, dx\right)^{1/2} \left(\int_a^b g(x)^2 \, dx\right)^{1/2} + \int_a^b g(x)^2 \, dx \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \|f\|^2_{L^2([a,b])} + 2\|f\|_{L^2([a,b])} \|g\|_{L^2([a,b])} + \|g\|^2_{L^2([a,b])} \\
&= \left(\|f\|_{L^2([a,b])} + \|g\|_{L^2([a,b])}\right)^2 .
\end{aligned}$$

Taking the square root gives the triangle inequality.

Remark: An alternative proof is to prove that the function $(\cdot,\cdot) : C([a,b]) \times C([a,b]) \to \mathbb{R}$,

$$(f,g) = \int_a^b f(x)g(x) \, dx,$$

is an inner product on $C([a,b])$. It then follows that $\|f\| := \sqrt{(f,f)}$ is a norm on $C([a,b])$ (the norm induced by the inner product; see Definition A.16 in the lecture notes). But this is just the L^2–norm $\|\cdot\|_{L^2([a,b])}$.

Remark: The Cauchy-Schwarz inequality can be proved by considering the quadratic polynomial

$$t \mapsto p(t) := \|f + tg\|^2_{L^2([a,b])}.$$

Since p is non-negative, then it must have non-positive discriminant, i.e., if $p(t) = \alpha t^2 + \beta t + \gamma$, then $\beta^2 - 4\alpha\gamma \le 0$. It is easy to check that this condition is exactly the Cauchy-Schwarz inequality.

(ii) We will prove that the function $(\cdot,\cdot)_{H^1} : C^1([a,b]) \times C^1([a,b]) \to \mathbb{R}$ defined by

$$(f,g)_{H^1} := \int_a^b f(x)g(x) \, dx + \int_a^b f'(x)g'(x) \, dx$$

is an inner product on $C^1([a,b])$. It then follows that

$$\|f\|_{H^1([a,b])} = \sqrt{(f,f)_{H^1}}$$

is a norm on $C^1([a,b])$ (see Definition A.16 in the lecture notes). It is clear that $(\cdot,\cdot)_{H^1}$ is symmetric and bilinear and that $(f,f)_{H^1} \ge 0$ for all $f \in C^1([a,b])$. Suppose that $(f,f)_{H^1} = 0$. Then $\|f\|_{H^1([a,b])} = 0$ and in particular $\|f\|_{L^2([a,b])} = 0$. Therefore $f = 0$ by part (i).

(iii) This is similar to part (ii). We will prove that the function $(\cdot,\cdot)_{H^1_0} : V \times V \to \mathbb{R}$ defined by

$$(f,g)_{H^1_0} := \int_a^b f'(x)g'(x) \, dx$$

is an inner product on V. It is clear that $(\cdot,\cdot)_{H^1_0}$ is symmetric and bilinear and that $(f,f)_{H^1_0} \ge 0$ for all $f \in V$. Suppose that $(f,f)_{H^1_0} = 0$. Then $\|f\|_{H^1_0([a,b])} = 0$ and in particular $\|f'\|_{L^2([a,b])} = 0$. Therefore $f' = 0$ by part (i) and so f is a constant function. But $f(a) = f(b) = 0$ and hence $f = 0$, as required.

(iv) We need to find constants $c, C > 0$ such that

$$c\|f\|_{H_0^1([a,b])} \leq \|f\|_{H^1([a,b])} \leq C\|f\|_{H_0^1([a,b])} \quad \forall f \in V.$$

Let $f \in V$. We have

$$\|f\|_{H_0^1([a,b])} = \|f'\|_{L^2([a,b])} \leq \left(\|f\|^2_{L^2([a,b])} + \|f'\|^2_{L^2([a,b])}\right)^{1/2} = \|f\|_{H^1([a,b])}.$$

Therefore $c = 1$. On the other hand,

$$\|f\|^2_{H^1([a,b])} = \|f\|^2_{L^2([a,b])} + \|f'\|^2_{L^2([a,b])} \leq C_{\mathrm{P}}^2\|f'\|^2_{L^2([a,b])} + \|f'\|^2_{L^2([a,b])}$$

where C_{P} is the Poincaré constant. Therefore we can take $C = (C_{\mathrm{P}}^2 + 1)^{1/2}$.

17. *Continuous dependence.* Let $u \in C^2(\overline{\Omega})$ satisfy

$$\begin{aligned} -\mathrm{div}(A\,\nabla u) + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Multiplying the PDE by u and integrating over Ω gives

$$\begin{aligned}
\int_\Omega fu\,d\boldsymbol{x} &= \int_\Omega u\,(-\mathrm{div}(A\,\nabla u) + cu)\,d\boldsymbol{x} \\
&= -\int_\Omega u\,\mathrm{div}(A\,\nabla u)\,d\boldsymbol{x} + c\int_\Omega u^2\,d\boldsymbol{x} \\
&= -\int_{\partial\Omega} u\,(A\,\nabla u)\cdot\boldsymbol{n}\,dS + \int_\Omega \nabla u\cdot(A\,\nabla u)\,d\boldsymbol{x} + c\int_\Omega u^2\,d\boldsymbol{x} \\
&= \int_\Omega (\nabla u)^{\mathrm{T}} A\,\nabla u\,d\boldsymbol{x} + c\int_\Omega u^2\,d\boldsymbol{x} && (u = 0 \text{ on } \partial\Omega) \\
&\geq \alpha\int_\Omega |\nabla u|^2\,d\boldsymbol{x} + c\int_\Omega u^2\,d\boldsymbol{x} && (A \text{ is uniformly positive definite}) \\
&\geq \min\{\alpha, c\}\left(\int_\Omega |\nabla u|^2\,d\boldsymbol{x} + \int_\Omega u^2\,d\boldsymbol{x}\right) \\
&= \min\{\alpha, c\}\,\|u\|^2_{H^1(\Omega)}
\end{aligned}$$

by definition of the H^1–norm. Therefore

$$\min\{\alpha, c\}\,\|u\|^2_{H^1(\Omega)} \leq \int_\Omega fu\,d\boldsymbol{x} \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$$

where we have used the Cauchy-Schwarz inequality and the fact that $\|v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)}$ for all $v \in C^1(\overline{\Omega})$. Cancelling one power of $\|u\|_{H^1(\Omega)}$ from both sides gives the desired result:

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

with $C = 1/\min\{\alpha, c\}$.

Remark: Note that this estimate degenerates as c tends to 0 ($C \to +\infty$ as $c \to 0$). If $c = 0$ or c is small then a better estimate can be obtained using the Poincaré inequality: As above

$$\int_\Omega fu\,d\boldsymbol{x} \geq \alpha\int_\Omega |\nabla u|^2\,d\boldsymbol{x} + c\int_\Omega u^2\,d\boldsymbol{x} \geq \alpha\int_\Omega |\nabla u|^2\,d\boldsymbol{x} = \alpha\|u\|^2_{H_0^1(\Omega)}.$$

Therefore

$$\alpha\|u\|^2_{H^1_0(\Omega)} \le \int_\Omega fu\,d\boldsymbol{x} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le C_{\mathrm{P}}\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} = C_{\mathrm{P}}\|f\|_{L^2(\Omega)}\|u\|_{H^1_0(\Omega)}$$

where $C_{\mathrm{P}}(\Omega)$ is the Poincaré constant. Cancelling one power of $\|u\|_{H^1_0(\Omega)}$ from both sides gives

$$\|u\|_{H^1_0(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

with $C = C_{\mathrm{P}}/\alpha$.

18. *Continuous dependence with a first-order term.*

(a) Let $u \in C^2(\overline{\Omega})$ satisfy

$$\begin{aligned} -k\Delta u + \boldsymbol{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

Multiply the PDE by u and integrate over Ω:

$$\begin{aligned} &-k\int_\Omega u\,\Delta u\,d\boldsymbol{x} + \int_\Omega u\,\boldsymbol{b}\cdot\nabla u\,d\boldsymbol{x} + \int_\Omega cu^2\,d\boldsymbol{x} = \int_\Omega fu\,d\boldsymbol{x} \\ \iff\quad & -k\left[\int_{\partial\Omega} u\,\nabla u\cdot\boldsymbol{n}\,dS - \int_\Omega |\nabla u|^2\,d\boldsymbol{x}\right] + \int_\Omega u\,\boldsymbol{b}\cdot\nabla u\,d\boldsymbol{x} + \int_\Omega cu^2\,d\boldsymbol{x} = \int_\Omega fu\,d\boldsymbol{x} \\ \iff\quad & k\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x} + c\|u\|^2_{L^2(\Omega)} = \int_\Omega fu\,d\boldsymbol{x} \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \end{aligned}$$

by the Cauchy-Schwarz inequality.

(b) Let $\varepsilon > 0$. Then

$$\begin{aligned} \left|\int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x}\right| &\le \|\boldsymbol{b}\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|\,|u|\,d\boldsymbol{x} \\ &\le \|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} && \text{(Cauchy-Schwarz)} \\ &= \|\boldsymbol{b}\|_{L^\infty(\Omega)}\left(\sqrt{2\varepsilon}\,\|\nabla u\|_{L^2(\Omega)}\right)\left(\frac{1}{\sqrt{2\varepsilon}}\,\|u\|_{L^2(\Omega)}\right) \\ &\le \|\boldsymbol{b}\|_{L^\infty(\Omega)}\left(\varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)}\right) \end{aligned}$$

by the Young inequality.

(c) Combining parts (a) and (b) gives

$$\begin{aligned} \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} &\ge k\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x} + c\|u\|^2_{L^2(\Omega)} \\ &\ge k\|\nabla u\|^2_{L^2(\Omega)} - \left|\int_\Omega (\boldsymbol{b}\cdot\nabla u)\,u\,d\boldsymbol{x}\right| + c\|u\|^2_{L^2(\Omega)} \\ &\ge k\|\nabla u\|^2_{L^2(\Omega)} - \|\boldsymbol{b}\|_{L^\infty(\Omega)}\left(\varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)}\right) + c\|u\|^2_{L^2(\Omega)} \\ &= \left(k - \varepsilon\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)\|\nabla u\|^2_{L^2(\Omega)} + \left(c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon}\right)\|u\|^2_{L^2(\Omega)}. \end{aligned}$$

(d) Let $\varepsilon > 0$ satisfy $k - \varepsilon \|\boldsymbol{b}\|_{L^\infty(\Omega)} > 0$, i.e., let

$$0 < \varepsilon < \frac{k}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}. \tag{20}$$

Let

$$c_0 = \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon}.$$

If $c > c_0$, then

$$c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon} > 0.$$

Therefore if $c > c_0$ and ε satisfies (20), then

$$k - \varepsilon \|\boldsymbol{b}\|_{L^\infty(\Omega)} > 0, \qquad c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon} > 0$$

and so by part (c)

$$\begin{aligned} \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} &\geq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\geq \min \left\{ k - \varepsilon \|\boldsymbol{b}\|_{L^\infty(\Omega)}, c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon} \right\} \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right) \\ &= \min \left\{ k - \varepsilon \|\boldsymbol{b}\|_{L^\infty(\Omega)}, c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon} \right\} \|u\|^2_{H^1(\Omega)}. \end{aligned}$$

Therefore if $c > c_0$ and ε satisfies (20), then

$$\|u\|_{H^1(\Omega)} \leq M \|f\|_{L^2(\Omega)}$$

with

$$M = \frac{1}{\min \left\{ k - \varepsilon \|\boldsymbol{b}\|_{L^\infty(\Omega)}, c - \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{4\varepsilon} \right\}}.$$

For example, if we choose

$$\varepsilon = \frac{1}{2} \frac{k}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}},$$

then

$$c_0 = \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{2k}, \qquad M = \frac{1}{\min \{k/2, c - c_0\}} = \max \left\{ \frac{2}{k}, \frac{1}{c - c_0} \right\}.$$

(e) Let $v \in C^2(\overline{\Omega})$ satisfy (19). Then $w = u - v$ satisfies (19) with $f = 0$. Therefore by part (d)

$$\|w\|_{H^1(\Omega)} \leq 0$$

and so $w = 0$ and $u = v$, as required.

19. *Neumann boundary conditions for variational problems.*

(i) Let $u \in C^1(\overline{\Omega})$ be a minimiser of E. For any $\varphi \in V$, $\varepsilon \in \mathbb{R}$, define $u_\varepsilon = u + \varepsilon\varphi$. Then $u_\varepsilon \in C^1(\overline{\Omega})$ since the sum of C^1 functions is C^1. Let $g(\varepsilon) = E[u_\varepsilon]$. Note that $u_\varepsilon = u$ when $\varepsilon = 0$. Therefore g is minimised by $\varepsilon = 0$ since E is minimised by u. Hence

$$
\begin{aligned}
0 = g'(0) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[u_\varepsilon] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \, d\boldsymbol{x} - \int_\Omega f u_\varepsilon \, d\boldsymbol{x}\right] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[\frac{1}{2}\int_\Omega (\nabla u + \varepsilon \nabla\varphi)\cdot(\nabla u + \varepsilon\nabla\varphi) \, d\boldsymbol{x} - \int_\Omega f(u+\varepsilon\varphi) \, d\boldsymbol{x}\right] \\
&= \frac{1}{2}\int_\Omega \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} [(\nabla u + \varepsilon \nabla\varphi)\cdot(\nabla u + \varepsilon\nabla\varphi)] \, d\boldsymbol{x} - \int_\Omega f \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (u+\varepsilon\varphi) \, d\boldsymbol{x} \\
&= \frac{1}{2}\int_\Omega \left.[\nabla\varphi\cdot(\nabla u + \varepsilon\nabla\varphi) + (\nabla u + \varepsilon\nabla\varphi)\cdot\nabla\varphi]\right|_{\varepsilon=0} d\boldsymbol{x} - \int_\Omega f\varphi \, d\boldsymbol{x} \\
&= \int_\Omega \nabla u \cdot \nabla\varphi \, d\boldsymbol{x} - \int_\Omega f\varphi \, d\boldsymbol{x}.
\end{aligned}
$$

Therefore

$$
\int_\Omega \nabla u \cdot \nabla \varphi \, d\boldsymbol{x} = \int_\Omega f\varphi \, d\boldsymbol{x} \quad \text{for all } \varphi \in C^1(\overline{\Omega}) \tag{21}
$$

as required.

(ii) First choose a test function $\varphi \in C^1(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$. Since $u \in C^2(\overline{\Omega})$, we can integrate by parts in (21) to obtain

$$
\int_{\partial\Omega} \nabla u \, \varphi \cdot \boldsymbol{n} \, dS - \int_\Omega \underbrace{\operatorname{div}\nabla u}_{=\Delta u} \varphi \, d\boldsymbol{x} = \int_\Omega f\varphi \, d\boldsymbol{x} \quad \iff \quad \int_\Omega (-\Delta u - f)\varphi \, d\boldsymbol{x} = 0
$$

because $\varphi = 0$ on $\partial\Omega$. Since this holds for all test functions $\varphi \in C^1(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$, the Fundamental Lemma of the Calculus of Variations implies that

$$
-\Delta u - f = 0 \quad \text{in } \Omega \tag{22}
$$

as required. We still need to show that u satisfies the Neumann boundary condition. Now take any test function $\varphi \in C^1(\overline{\Omega})$ in (21) and integrate by parts as before to obtain

$$
\begin{aligned}
\int_{\partial\Omega} \nabla u \, \varphi \cdot \boldsymbol{n} \, dS - \int_\Omega \Delta u \, \varphi \, d\boldsymbol{x} = \int_\Omega f\varphi \, d\boldsymbol{x} \quad &\iff \quad \int_{\partial\Omega} \nabla u \, \varphi \cdot \boldsymbol{n} \, dS + \int_\Omega \underbrace{(-\Delta u - f)}_{=0 \text{ by (22)}} \varphi \, d\boldsymbol{x} = 0 \\
&\iff \quad \int_{\partial\Omega} \nabla u \cdot \boldsymbol{n} \, \varphi \, dS = 0.
\end{aligned}
$$

Since this holds for all $\varphi \in C^1(\overline{\Omega})$, then $\nabla u \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, as required.

20. *The p–Laplacian operator.*

(i) Let $u \in C^2(\overline{\Omega}) \cap V$ minimise E_p. For any $\varphi \in V$, $\varepsilon \in \mathbb{R}$, define $u_\varepsilon = u + \varepsilon\varphi$. Observe that u_ε vanishes on the boundary of Ω since both u and φ vanish there. Also $u_\varepsilon \in C^1(\overline{\Omega})$ since the sum of C^1 functions is C^1. Hence $u_\varepsilon \in V$. Define $g(\varepsilon) = E_p[u_\varepsilon]$. Now $u_\varepsilon = u$ when $\varepsilon = 0$. Therefore g is minimised by $\varepsilon = 0$ since E_p is minimised by u. We have reduced the problem of minimising the functional E_p to minimising the function of one variable g. Since g is minimised at $\varepsilon = 0$,

$$\begin{aligned}
0 &= g'(0) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_p[u_\varepsilon] \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left[\frac{1}{p}\int_\Omega |\nabla u_\varepsilon|^p \, d\boldsymbol{x} - \int_\Omega f u_\varepsilon \, d\boldsymbol{x}\right] \\
&= \frac{1}{p}\int_\Omega \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} |\nabla u + \varepsilon\nabla\varphi|^p \, d\boldsymbol{x} - \int_\Omega f \, \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (u + \varepsilon\varphi)\, d\boldsymbol{x} \\
&= \frac{1}{p}\int_\Omega p|\nabla u + \varepsilon\nabla\varphi|^{p-1} \frac{\nabla u + \varepsilon\nabla\varphi}{|\nabla u + \varepsilon\nabla\varphi|} \cdot \nabla\varphi\bigg|_{\varepsilon=0} d\boldsymbol{x} - \int_\Omega f\varphi \, d\boldsymbol{x} \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, d\boldsymbol{x} - \int_\Omega f\varphi \, d\boldsymbol{x}
\end{aligned} \tag{23}$$

where the differentiation was performed using the Chain Rule and the fact that

$$\frac{d}{dx}\, x^p = p x^{p-1}, \qquad \nabla_{\boldsymbol{y}} \, |\boldsymbol{y}| = \frac{\boldsymbol{y}}{|\boldsymbol{y}|}, \qquad \frac{d}{d\varepsilon}(\nabla u + \varepsilon\nabla\varphi) = \nabla\varphi.$$

Recall the integration by parts formula

$$\int_\Omega \boldsymbol{g} \cdot \nabla h \, d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{g} h \cdot \boldsymbol{n} \, dS - \int_\Omega h \operatorname{div} \boldsymbol{g} \, d\boldsymbol{x}.$$

By applying this with $h = \varphi$, $\boldsymbol{g} = |\nabla u|^{p-2}\nabla u$, we can rewrite equation (23) as

$$0 = \int_{\partial\Omega} |\nabla u|^{p-2}\nabla u \, \varphi \cdot \boldsymbol{n} \, dS - \int_\Omega \varphi \operatorname{div}(|\nabla u|^{p-2}\nabla u) \, d\boldsymbol{x} - \int_\Omega f\varphi \, d\boldsymbol{x}.$$

But $\varphi = 0$ on $\partial\Omega$ since $\varphi \in V$. Therefore

$$0 = \int_\Omega [\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f] \, \varphi \, d\boldsymbol{x} \quad \text{for all } \varphi \in V.$$

Since φ is arbitrary, the Fundamental Lemma of the Calculus of Variations gives

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f = 0 \quad \text{in } \Omega.$$

Therefore

$$-\underbrace{\operatorname{div}(|\nabla u|^{p-2}\nabla u)}_{=\Delta_p u} = f \quad \text{in } \Omega$$

as required. Note that $u = 0$ on $\partial\Omega$ by definition of V.

(ii) Multiply the PDE $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ by u and integrate by parts over Ω to obtain

$$\begin{aligned}
& -\int_\Omega u \operatorname{div}(|\nabla u|^{p-2}\nabla u) \, d\boldsymbol{x} = \int_\Omega f u \, d\boldsymbol{x} \\
\iff \quad & -\int_{\partial\Omega} u(|\nabla u|^{p-2}\nabla u) \cdot \boldsymbol{n} \, dS + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u \, d\boldsymbol{x} = \int_\Omega f u \, d\boldsymbol{x} \\
\iff \quad & \int_\Omega |\nabla u|^p \, d\boldsymbol{x} = \int_\Omega f u \, d\boldsymbol{x}
\end{aligned} \tag{24}$$

since $u = 0$ on $\partial\Omega$. Therefore

$$\begin{aligned}
E_p[u] &= \frac{1}{p}\int_\Omega |\nabla u|^p \, d\boldsymbol{x} - \int_\Omega f u \, d\boldsymbol{x} \\
&= \frac{1}{p}\int_\Omega |\nabla u|^p \, d\boldsymbol{x} - \int_\Omega |\nabla u|^p \, d\boldsymbol{x} && \text{(by equation (24))} \\
&= \frac{1-p}{p}\int_\Omega |\nabla u|^p \, d\boldsymbol{x} \\
&= \frac{1-p}{p}\int_\Omega f u \, d\boldsymbol{x} && \text{(by equation (24))}
\end{aligned}$$

as required.

21. *The minimal surface equation: PDEs and soap films.* Let $u \in C^2(\overline{\Omega}) \cap V$ be a minimiser of A. Let $\varepsilon \in \mathbb{R}$ and $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$ on $\partial\Omega$. Define $u_\varepsilon = u + \varepsilon\varphi$. Then $u_\varepsilon \in V$ since the sum of continuously differential functions is continuously differentiable and, if $\boldsymbol{x} \in \partial\Omega$, then

$$u_\varepsilon(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon\varphi(\boldsymbol{x}) = g(\boldsymbol{x}) + \varepsilon \cdot 0 = g(\boldsymbol{x})$$

as required. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(\varepsilon) = A[u_\varepsilon]$. Then $h(0) = A[u]$ and so 0 is a minimum point of h since u is a minimum point of A. Therefore

$$\begin{aligned}
0 &= h'(0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A[u_\varepsilon] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega \sqrt{1 + |\nabla u_\varepsilon|^2} \, d\boldsymbol{x} \\
&= \int_\Omega \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \sqrt{1 + |\nabla u + \varepsilon\nabla\varphi|^2} \, d\boldsymbol{x} \\
&= \int_\Omega \left. \tfrac{1}{2}(1 + |\nabla u + \varepsilon\nabla\varphi|^2)^{-1/2} \, 2|\nabla u + \varepsilon\nabla\varphi| \, \frac{\nabla u + \varepsilon\nabla\varphi}{|\nabla u + \varepsilon\nabla\varphi|} \cdot \nabla\varphi \right|_{\varepsilon=0} d\boldsymbol{x} \\
&= \int_\Omega \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla\varphi \, d\boldsymbol{x}.
\end{aligned}$$

This means that u is a weak solution of the minimal surface equation. Since $u \in C^2(\overline{\Omega})$, then we can integrate by parts to obtain

$$\begin{aligned}
0 &= \int_\Omega \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nabla\varphi \, d\boldsymbol{x} \\
&= \int_{\partial\Omega} \varphi \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \boldsymbol{n} \, dS - \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \varphi \, d\boldsymbol{x} \\
&= -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \varphi \, d\boldsymbol{x}
\end{aligned}$$

since $\varphi = 0$ on $\partial\Omega$. This holds for all $\varphi \in C^1(\overline{\Omega})$ with $\varphi = 0$. Therefore u satisfies the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{in } \Omega.$$

by the Fundamental Lemma of the Calculus of Variations (Lemma 3.20).

22. *Homogenization and the calculus of variations.*

(i) Let $u \in C^2([0,1]) \cap V$ minimise E. For any $\varepsilon \in \mathbb{R}$ and any $\varphi \in C^1([0,1])$ such that $\varphi(0) = \varphi(1) = 0$, define $u_\varepsilon = u + \varepsilon\varphi$. Then

$$u_\varepsilon(0) = u(0) + \varepsilon\varphi(0) = l + \varepsilon \cdot 0 = l$$

and similarly $u_\varepsilon(1) = r$. Therefore $u_\varepsilon \in V$. Define $F(\varepsilon) = E[u_\varepsilon]$. Now $u_\varepsilon = u$ when $\varepsilon = 0$. Therefore the minimum of F is attained at 0 since the minimum of E is attained at u. We have reduced the problem of minimising the functional E to minimising the function of one variable F. Since F is minimised at 0,

$$\begin{aligned}
0 &= F'(0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E[u_\varepsilon] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[\frac{1}{2}\int_0^1 a(x)|u_\varepsilon'(x)|^2\,dx - \int_0^1 f(x)u_\varepsilon(x)\,dx\right] \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[\frac{1}{2}\int_0^1 a(x)(u'(x)+\varepsilon\varphi'(x))^2\,dx - \int_0^1 f(x)(u(x)+\varepsilon\varphi(x))\,dx\right] \\
&= \int_0^1 a(x)u'(x)\varphi'(x)\,dx - \int_0^1 f(x)\varphi(x)\,dx. \qquad (25)
\end{aligned}$$

Since $u \in C^2([0,1])$, we can use integration by parts to rewrite equation (25) as

$$0 = a(x)u'(x)\varphi(x)\Big|_0^1 - \int_0^1 (a(x)u'(x))'\varphi(x)\,dx - \int_0^1 f(x)\varphi(x)\,dx = -\int_0^1 [(a(x)u'(x))' + f(x)]\varphi(x)\,dx.$$

But this holds for all $\varphi \in C^1([0,1])$ such that $\varphi(0) = \varphi(1) = 0$. Therefore by the Fundamental Lemma of the Calculus of Variations

$$(a(x)u'(x))' + f(x) = 0, \quad x \in (0,1),$$

as required. Note that u satisfies the Dirichlet boundary conditions by definition of V.

(ii) Recall from Q2(ii) that if $g \in L^\infty(\mathbb{R})$ is 1–periodic, then for any interval $[c,d] \subseteq \mathbb{R}$,

$$\lim_{n\to\infty} \int_c^d g(nx)h(x)\,dx = \int_c^d \overline{g}\,h(x)\,dx \qquad \forall\, h \in L^1(\mathbb{R}). \qquad (26)$$

Applying (26) with $c = 0$, $d = 1$, $g(x) = a(x)$, $h(x) = \frac{1}{2}|v'(x)|^2$ on $[0,1]$, gives the desired result:

$$\lim_{n\to\infty} E_n[v] = \frac{1}{2}\int_0^1 \overline{a}\,|v'(x)|^2\,dx - \int_0^1 f(x)v(x)\,dx =: E_\infty[v].$$

(iii) Observe that E_∞ is just the one-dimensional Dirichlet energy with an additional constant $\overline{a}$ in the first term. It follows from Dirichlet's Principle (see the lecture notes) that u_∞ satisfies the Poisson equation

$$\begin{aligned}
-\overline{a}\,u_\infty''(x) &= f(x), \quad x \in (0,1), \\
u_\infty(0) &= u_\infty(1) = 0.
\end{aligned}$$

In Q2 we showed that $\lim_{n\to\infty} u_n(x) = u_0(x)$, where u_0 satisfies

$$
\begin{aligned}
-a_0 u_0''(x) &= f(x), \quad x \in (0,1), \\
u_0(0) &= u_0(1) = 0,
\end{aligned}
$$

where

$$
a_0 = \frac{1}{\overline{\left(\frac{1}{a}\right)}}.
$$

Since $a_0 \neq \overline{a}$ in general, it follows that $u_0 \neq u_\infty$ and hence

$$
\lim_{n \to \infty} u_n(x) = u_0(x) \neq u_\infty(x),
$$

as required.

In fact it can be shown that $a_0 \leq \overline{a}$ as follows:

$$
1 = \left[\int_0^1 \sqrt{a(x)} \frac{1}{\sqrt{a(x)}} \, dx \right]^2 \leq \left[\left(\int_0^1 a(x) \, dx \right)^{1/2} \left(\int_0^1 \frac{1}{a(x)} \, dx \right)^{1/2} \right]^2 = \overline{a} \, \overline{\left(\frac{1}{a} \right)} = \overline{a} \, a_0^{-1}
$$

where we have used the Cauchy-Schwarz inequality. It follows that the Γ–limit E_0 is less than or equal to the pointwise limit E_∞.