Partial Differential Equations III/IV
Exercise Sheet 5

1. *Mean-value formula* $\implies$ *harmonic.* Let $\Omega \subseteq \mathbb{R}^2$ be open. Let $u \in C^2(\Omega)$ satisfy the mean-value formula

$$u(\boldsymbol{x}) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y})$$

for all balls $B_r(\boldsymbol{x}) \subset \Omega$. Show that u is harmonic in Ω.
Hint: The proof is very similar to the proof of the converse statement.
Remark: It is actually enough to assume only that $u \in C(\Omega)$. From the solution of Q10, we see that if $u \in C(\Omega)$ satisfies the mean-value formula, then $u \in C^\infty(\Omega)$. Therefore u is harmonic by Q1.

2. *Subharmonic functions.* Let $\Omega \subset \mathbb{R}^2$ be open, bounded and connected. We say that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic in Ω if

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

   (i) Prove that subharmonic functions satisfy the mean-value formula

$$u(\boldsymbol{x}) \leq \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y}$$

   for all balls $B_r(\boldsymbol{x}) \subset \Omega$.

   (ii) Prove that subharmonic functions satisfy the maximum principle

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

   (iii) Do subharmonic functions satisfy the minimum principle

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u\ ?$$

   Hint: Think about the one-dimensional case.

3. *Strong maximum principle* $\implies$ *weak maximum principle.* Use the strong maximum principle for Laplace's equation to prove the weak maximum principle. To be precise, let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω. The strong maximum principle asserts that if u attains its maximum in Ω, then u is constant, i.e., if there exists $\boldsymbol{x}_0 \in \Omega$ such that

$$u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u$$

then u is constant in Ω. Use this to prove the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

4. *The strong maximum principle is false if* Ω *is not connected.* Find an example of an open, bounded and *disconnected* set $\Omega \subset \mathbb{R}^2$ and a *non-constant* harmonic function $u : \overline{\Omega} \to \mathbb{R}$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u$$

for some $\boldsymbol{x}_0 \in \Omega$.
Remark: The weak maximum principle, on the other hand, does hold on disconnected sets.

5. *Minimum principles and an application: Positivity of solutions.*

   (i) Use the maximum principles for harmonic functions to prove the corresponding minimum principles. Hint: If u is harmonic, so is $\tilde{u} = -u$. Apply the maximum principles to $\tilde{u}$.

   (ii) Let $\Omega \subset \mathbb{R}^2$ be open, bounded and connected and let $g \in C(\partial\Omega)$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

   Assume that $g(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \partial\Omega$ and that there exists $\boldsymbol{x}_0 \in \partial\Omega$ such that $g(\boldsymbol{x}_0) > 0$. Prove the positivity result

$$u(\boldsymbol{x}) > 0 \quad \text{for all } \boldsymbol{x} \in \Omega.$$

6. *Another application of the maximum principle: Bounds on solutions.* In class we used the maximum principle to prove uniqueness for Poisson's equation. Another application is to prove bounds on solutions, as this question demonstrates. This question appeared on the May 2010 exam.

   (a) If u is harmonic in $|x| < 1$, $|y| < 1$, and $u = x^2 + y^2$ on the boundary lines $|x| = 1$ and $|y| = 1$, find lower and upper bounds for $u(0,0)$.

   (b) Verify that

$$v = \frac{47}{40} - \frac{1}{5}(x^4 - 6x^2y^2 + y^4)$$

   is harmonic and that $-0.025 \leq v - 1 - x^2 \leq 0.025$ when $|x| < 1$ and $|y| = 1$. Deduce that $u(0,0)$ of Part (a) satisfies $1.15 < u(0,0) < 1.2$.

7. *Application of the maximum principle for subharmonic functions: Comparison theorems.* Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. For $i \in \{1,2\}$, let $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u_i &= f_i \quad \text{in } \Omega, \\ u_i &= g_i \quad \text{on } \partial\Omega, \end{aligned}$$

   where $f_i \in C(\Omega)$, $g_i \in C(\partial\Omega)$, $i \in \{1,2\}$. Assume that $f_1 \leq f_2$ and $g_1 \leq g_2$. Prove that $u_1 \leq u_2$. This is know as a *comparison theorem* or a *comparison principle*.

8. *Maximum principles for more general elliptic problems.* Maximum principles hold not only for Laplace's equation, but also for a broad class of second-order linear elliptic PDEs. In this exercise we look at some examples.

   (i) Consider the one-dimensional steady convection-diffusion equation

$$-\alpha u'' + \beta u' = 0 \quad \text{in } (a,b)$$

   where α and β are constants, $\alpha > 0$. Show that u satisfies a maximum and minimum principle.

   (ii) Consider Poisson's equation

$$-u'' = f \quad \text{in } (a,b)$$

   where f is a constant. Under what conditions on f does u satisfy a maximum principle? And a minimum principle?

   (iii) Consider the equation

$$-u'' + cu = 0 \quad \text{in } \Omega$$

   with $c > 0$, $\Omega = (a,b)$. Show that $\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$. What if $c < 0$?

9. *Maximum principles for 4th-order elliptic PDEs?* Do 4th-order elliptic PDEs satisfy a maximum principle? Think about the differential equations $u'''' = 0$ and $u'''' = f$, where f is a constant.

10. *Regularity Theorem: Harmonic functions are C^∞.* Let $\Omega \subseteq \mathbb{R}^2$ be open. We will prove that if $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$.

    (i) Define $\eta \in C^\infty(\mathbb{R}^2)$ by

    $$\eta(\boldsymbol{x}) = \begin{cases} C \exp\left(-\frac{1}{1-|\boldsymbol{x}|^2}\right) & \text{if } |\boldsymbol{x}| < 1, \\ 0 & \text{if } |\boldsymbol{x}| \geq 1, \end{cases}$$

    where C is the normalisation constant

    $$C = \left(\int_{B_1(\boldsymbol{0})} e^{-\frac{1}{1-|\boldsymbol{x}|^2}} \, d\boldsymbol{x} \right)^{-1}.$$

    For $\varepsilon > 0$, define

    $$\eta_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2} \eta\left(\frac{\boldsymbol{x}}{\varepsilon}\right).$$

    Find the support of η_ε, $\mathrm{supp}(\eta_\varepsilon)$. Show that for all $\varepsilon > 0$

    $$\int_{\mathbb{R}^2} \eta_\varepsilon(\boldsymbol{x}) \, d\boldsymbol{x} = 1.$$

    (ii) Define

    $$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega \,:\, B_\varepsilon(\boldsymbol{x}) \subset \Omega\} = \{\boldsymbol{x} \in \Omega \,:\, \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

    For $\boldsymbol{x} \in \Omega_\varepsilon$, define

    $$u_\varepsilon(\boldsymbol{x}) = \int_\Omega \eta_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y} = \int_{B_\varepsilon(\boldsymbol{x})} \eta_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y}.$$

    (You should recognise this as a type of convolution.) Observe that

    $$\frac{\partial u_\varepsilon}{\partial x_i}(\boldsymbol{x}) = \int_\Omega \frac{\partial \eta_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y}$$

    and similarly for higher-order derivatives. It follows that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ since η_ε is infinitely differentiable. Use the mean-value formula

    $$u(\boldsymbol{x}) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dL(\boldsymbol{y})$$

    to prove that

    $$u_\varepsilon(\boldsymbol{x}) = u(\boldsymbol{x})$$

    for all $\boldsymbol{x} \in \Omega_\varepsilon$. Therefore $u \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ and hence $u \in C^\infty(\Omega)$.
    Hint: Use polar coordinates to rewrite the formula for $u_\varepsilon(\boldsymbol{x})$ in terms of

    $$\int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dL(\boldsymbol{y}).$$

    You will also need to use part (i) and the fact that η_ε is a radial function, which means that $\eta_\varepsilon(\boldsymbol{x})$ depends only on $|\boldsymbol{x}|$.

11. $C^\infty \not\Longrightarrow$ *analytic.* Harmonic functions are analytic, which means that they are infinitely differentiable and that they have a convergent Taylor series expansion about every point in their domain. Give an example of an infinitely differentiable function that is not analytic.
Hint: Can nonzero analytic functions have compact support?

12. *Non-negative harmonic functions on $\mathbb{R}^n$ are constant.* Let $u : \mathbb{R}^n \to [0, \infty)$ be harmonic and non-negative.

   (i) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $R > r > 0$, and $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$. Use a mean-value formula to prove that
$$u(\boldsymbol{x}) \le \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} u(\boldsymbol{y}).$$
Hint: Write
$$u(\boldsymbol{x}) = \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{z})\, d\boldsymbol{z} = \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} \frac{1}{|B_R(\boldsymbol{y})|} \int_{B_r(\boldsymbol{x})} u(\boldsymbol{z})\, d\boldsymbol{z}.$$

   (ii) Choose $r = R - |\boldsymbol{x} - \boldsymbol{y}|$. Show that $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$ and compute
$$\lim_{R \to \infty} \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|}.$$

   (iii) Conclude that u is constant.

13. *Proof of Liouville's Theorem.* Use the previous question to prove Liouville's Theorem.

14. *An application of Liouville's Theorem: 'Uniqueness' for Poisson's equation in $\mathbb{R}^3$.* Let $f \in C_c^2(\mathbb{R}^3)$. Let $u \in C^2(\mathbb{R}^3)$ be a *bounded* solution of Poisson's equation in $\mathbb{R}^3$:
$$-\Delta u = f \quad \text{in } \mathbb{R}^3.$$
Prove that
$$u = \Phi * f + c$$
for some constant $c \in \mathbb{R}$, where Φ is the fundamental solution of Poisson's equation in $\mathbb{R}^3$. This means that bounded solutions of Poisson's equation in $\mathbb{R}^3$ are unique up to a constant.
Hint: Let $u_1 = \Phi * f$ and let u_2 be any bounded solution of Poisson's equation in $\mathbb{R}^3$. Apply Liouville's Theorem to $w = u_2 - u_1$. The same argument works in $\mathbb{R}^n$ for $n \ge 3$. Why doesn't this argument work in $\mathbb{R}^2$?

15. *An obstacle to uniqueness for Laplace's equation: Unbounded domains.* Let $n > 1$ and let $\Omega \subset \mathbb{R}^n$ be open. Consider the PDE
$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$
Clearly $u = 0$ is one solution of (1). Find a nontrivial solution of (1) for

   (i) $\Omega = \mathbb{R}^n \setminus \overline{B_1(\mathbf{0})}$;
(hint: consider the fundamental solution of Poisson's equation)

   (ii) $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : x_n > 0\}$.

These examples are taken from Q. Han (2011) *A Basic Course in Partial Differential Equations*, AMS.

16. *Eigenvalues of the negative Laplacian.* Consider the eigenvalue problem

$$-u''(x) = \lambda u(x), \quad x \in (0, 2\pi),$$
$$u(0) = 0, \; u(2\pi) = 0,$$

where $u \in C^2([0, 2\pi])$, $u \neq 0$, $\lambda \in \mathbb{R}$. Show that there are countably many eigenfunction-eigenvalue pairs (u_n, λ_n), $n \in \mathbb{N}$, and find them all.
Hint: We know from Exercise Sheet 4, Q11, that all the eigenvalues are positive. Therefore we can assume that $\lambda = \omega^2$ from some $\omega \in (0, \infty)$. We also know the general form of solutions of the second-order linear ODE $u''(x) + \omega^2 u(x) = 0$; see the lecture notes, page 21.
Remark: More generally, it can be shown that second-order linear elliptic differential operators on compact sets have a countable set of eigenvalues.

17. *Connection between holomorphic functions and harmonic functions.* Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic (complex analytic) function with real and imaginary parts u and v:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Use the Cauchy-Riemann equations to show that u and v are harmonic functions. Several results for holomorphic functions can be extended to the broader class of harmonic functions. Complete the following table to give the names of the analogous results in complex analysis:

Harmonic Functions	Holomorphic Functions
Mean-Value Formula	
Maximum Principle	
Liouville's Theorem	

The regularity result for harmonic functions (if $u \in C^2$ is harmonic, then u is analytic) also has an analogue for holomorphic functions: If f is complex differentiable, then f is complex analytic.