Partial Differential Equations III/IV
Exercise Sheet 5: Solutions

1. *Mean-value formula* $\implies$ *harmonic.* Fix $\boldsymbol{x} \in \Omega$. For all $B_r(\boldsymbol{x}) \subset \Omega$

$$u(\boldsymbol{x}) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y}) =: \phi(r). \tag{1}$$

We can parametrise $\partial B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^2 : |\boldsymbol{y} - \boldsymbol{x}| = r\}$ using polar coordinates by

$$\boldsymbol{r} : [0, 2\pi] \to \partial B_r(\boldsymbol{x}), \quad \boldsymbol{r}(\theta) = \boldsymbol{x} + r(\cos\theta, \sin\theta).$$

Using this parametrisation we compute

$$\begin{aligned}
\phi(r) &= \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y}) = \frac{1}{|\partial B_r(\boldsymbol{x})|} \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y}) \\
&= \frac{1}{2\pi r} \int_0^{2\pi} u(\boldsymbol{r}(\theta)) |\dot{\boldsymbol{r}}(\theta)|\, d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta))\, r\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta))\, d\theta. \qquad (2)
\end{aligned}$$

By equation (2) and the Chain Rule

$$\begin{aligned}
\phi'(r) &= \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta))\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \nabla u(\boldsymbol{x} + r(\cos\theta, \sin\theta)) \cdot (\cos\theta, \sin\theta)\, d\theta. \qquad (3)
\end{aligned}$$

The unit outward-pointing normal to $\partial B_r(\boldsymbol{x})$ at point $\boldsymbol{y}$ is

$$\boldsymbol{n}(\boldsymbol{y}) = \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|} = \frac{\boldsymbol{y} - \boldsymbol{x}}{r}.$$

Taking $\boldsymbol{y} = \boldsymbol{r}(\theta)$ gives

$$\boldsymbol{n}(\boldsymbol{r}(\theta)) = \frac{\boldsymbol{r}(\theta) - \boldsymbol{x}}{r} = (\cos\theta, \sin\theta).$$

Using this, we can write equation (3) as

$$\begin{aligned}
\phi'(r) &= \frac{1}{2\pi} \int_0^{2\pi} \nabla u(\boldsymbol{r}(\theta)) \cdot \boldsymbol{n}(\boldsymbol{r}(\theta))\, d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} \nabla u(\boldsymbol{r}(\theta)) \cdot \boldsymbol{n}(\boldsymbol{r}(\theta)) \underbrace{r}_{=|\dot{\boldsymbol{r}}(\theta)|}\, d\theta \\
&= \frac{1}{2\pi r} \int_{\partial B_r(\boldsymbol{x})} \nabla u(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, dL(\boldsymbol{y}) \\
&= \frac{1}{2\pi r} \int_{B_r(\boldsymbol{x})} \operatorname{div} \nabla u(\boldsymbol{y})\, d\boldsymbol{y} \qquad \text{(Divergence Theorem)} \\
&= \frac{1}{2\pi r} \int_{B_r(\boldsymbol{x})} \Delta u(\boldsymbol{y})\, d\boldsymbol{y}.
\end{aligned}$$

Differentiating equation (1) with respect to r gives $\phi'(r) = 0$. Therefore

$$0 = \phi'(r) = \frac{1}{2\pi r} \int_{B_r(\boldsymbol{x})} \Delta u(\boldsymbol{y}) \, d\boldsymbol{y}$$

for all $B_r(\boldsymbol{x}) \subset \Omega$. There are two ways to reach the punchline from here: Either observe that since

$$\int_{B_r(\boldsymbol{x})} \Delta u(\boldsymbol{y}) \, d\boldsymbol{y} = 0 \quad \forall \, B_r(\boldsymbol{x}) \subset \Omega, \tag{4}$$

then we must have $\Delta u(\boldsymbol{x}) = 0$. (Otherwise, by continuity of Δu, Δu is either strictly positive or strictly negative in $B_r(\boldsymbol{x})$ for r sufficiently small, which contradicts (4).) Alternatively, multiply equation (4) by $\frac{1}{\pi r^2}$ and take the limit $r \to 0$:

$$\frac{1}{\pi r^2} \int_{B_r(\boldsymbol{x})} \Delta u(\boldsymbol{y}) \, d\boldsymbol{y} = 0 \quad \overset{r \to 0}{\Longrightarrow} \quad \Delta u(\boldsymbol{x}) = 0$$

since the average of a continuous function over a ball of radius r tends to the value of the function at the centre of the ball as $r \to 0$.

2. *Subharmonic functions.*

   (i) First we prove the mean-value formula

$$u(\boldsymbol{x}) \leq \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dL(\boldsymbol{y}) =: \phi(r). \tag{5}$$

We can parametrise $\partial B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^2 : |\boldsymbol{y} - \boldsymbol{x}| = r\}$ using polar coordinates by

$$\boldsymbol{r} : [0, 2\pi] \to \partial B_r(\boldsymbol{x}), \quad \boldsymbol{r}(\theta) = \boldsymbol{x} + r(\cos\theta, \sin\theta).$$

Using this parametrisation we compute

$$\begin{aligned}
\phi(r) = \fint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dL(\boldsymbol{y}) &= \frac{1}{|\partial B_r(\boldsymbol{x})|} \int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dL(\boldsymbol{y}) \\
&= \frac{1}{2\pi r} \int_0^{2\pi} u(\boldsymbol{r}(\theta)) |\dot{\boldsymbol{r}}(\theta)| \, d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta)) \, r \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta)) \, d\theta.
\end{aligned} \tag{6}$$

By equation (6) and the Chain Rule

$$\begin{aligned}
\phi'(r) &= \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta)) \, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \nabla u(\boldsymbol{x} + r(\cos\theta, \sin\theta)) \cdot (\cos\theta, \sin\theta) \, d\theta.
\end{aligned} \tag{7}$$

The unit outward-pointing normal to $\partial B_r(\boldsymbol{x})$ at point $\boldsymbol{y}$ is

$$\boldsymbol{n}(\boldsymbol{y}) = \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|} = \frac{\boldsymbol{y} - \boldsymbol{x}}{r}.$$

Taking $\boldsymbol{y} = \boldsymbol{r}(\theta)$ gives

$$\boldsymbol{n}(\boldsymbol{r}(\theta)) = \frac{\boldsymbol{r}(\theta) - \boldsymbol{x}}{r} = (\cos\theta, \sin\theta).$$

Using this, we can write equation (7) as

$$\begin{aligned}
\phi'(r) &= \frac{1}{2\pi} \int_0^{2\pi} \nabla u(\boldsymbol{r}(\theta)) \cdot \boldsymbol{n}(\boldsymbol{r}(\theta))\, d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} \nabla u(\boldsymbol{r}(\theta)) \cdot \boldsymbol{n}(\boldsymbol{r}(\theta)) \underbrace{r}_{=|\dot{\boldsymbol{r}}(\theta)|} d\theta \\
&= \frac{1}{2\pi r} \int_{\partial B_r(\boldsymbol{x})} \nabla u(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})\, dL(\boldsymbol{y}) \\
&= \frac{1}{2\pi r} \int_{B_r(\boldsymbol{x})} \operatorname{div} \nabla u(\boldsymbol{y})\, d\boldsymbol{y} && \text{(Divergence Theorem)} \\
&= \frac{1}{2\pi r} \int_{B_r(\boldsymbol{x})} \underbrace{\Delta u(\boldsymbol{y})}_{\geq 0}\, d\boldsymbol{y} \\
&\geq 0
\end{aligned}$$

since u is subharmonic. Therefore $\phi'(r) \geq 0$ and hence $\phi(r) \geq \phi(0)$ if $r \geq 0$. The mean-value formula (5) follows almost immediately from this:

$$\phi(r) \geq \phi(0) = \lim_{r\to 0} \phi(r) = \lim_{r \to 0} ⨍_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y}) = u(\boldsymbol{x})$$

since the average of a continuous function over a sphere of radius r tends to the value of the function at the centre of the sphere as $r \to 0$.

Now we prove the second mean-value formula

$$u(\boldsymbol{x}) \leq ⨍_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y}. \tag{8}$$

Using polar coordinates we can write

$$\begin{aligned}
⨍_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} &= \frac{1}{\pi r^2} \int_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \frac{1}{\pi r^2} \int_{\rho=0}^{r} \int_{\theta=0}^{2\pi} u(\boldsymbol{x} + \rho(\cos\theta, \sin\theta))\, \rho\, d\theta d\rho.
\end{aligned} \tag{9}$$

Observe that $\partial B_\rho(\boldsymbol{x})$ is parametrised by $\boldsymbol{r}_\rho : [0, 2\pi] \to \partial B_\rho(\boldsymbol{x})$, $\boldsymbol{r}_\rho(\theta) = \boldsymbol{x} + \rho(\cos\theta, \sin\theta)$. This parametrisation satisfies $|\dot{\boldsymbol{r}}_\rho| = \rho$. Therefore we can write equation (9) as

$$\begin{aligned}
⨍_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} &= \frac{1}{\pi r^2} \int_{\rho=0}^{r} \int_{\theta=0}^{2\pi} u(\boldsymbol{r}_\rho(\theta))\, |\dot{\boldsymbol{r}}_\rho|\, d\theta d\rho \\
&= \frac{1}{\pi r^2} \int_{\rho=0}^{r} \underbrace{\left(\int_{\partial B_\rho(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y}) \right)}_{\geq 2\pi\rho\, u(\boldsymbol{x}) \text{ by } (5)} d\rho \\
&\geq \frac{u(\boldsymbol{x})}{r^2} \int_{\rho=0}^{r} 2\rho\, d\rho \\
&= \frac{u(\boldsymbol{x})}{r^2} \rho^2 \Big|_0^r \\
&= u(\boldsymbol{x})
\end{aligned}$$

as required.

(ii) We prove the strong maximum principle. Let $\boldsymbol{x}_0 \in \Omega$ and

$$M = u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u.$$

Define S to be the set of points in Ω where u attains its maximum:

$$S = \{\boldsymbol{x} \in \Omega \,:\, u(\boldsymbol{x}) = M\} = u^{-1}(\{M\}) \cap \Omega.$$

Note that S is nonempty since $\boldsymbol{x}_0 \in S$.
Let $\boldsymbol{x} \in S$ and $B_r(\boldsymbol{x}) \subset \Omega$, i.e, $0 < r < \mathrm{dist}(\boldsymbol{x}, \partial\Omega)$. By part (i)

$$M = u(\boldsymbol{x}) \leq \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} \leq \fint_{B_r(\boldsymbol{x})} M\, d\boldsymbol{y} = M. \tag{10}$$

Therefore the inequality in (10) is an equality,

$$\fint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, d\boldsymbol{y} = \fint_{B_r(\boldsymbol{x})} M\, d\boldsymbol{y},$$

which means that $u(\boldsymbol{y}) = M$ for all $\boldsymbol{y} \in B_r(\boldsymbol{x})$. Hence $B_r(\boldsymbol{x}) \subset S$ and so S is an open subset of Ω.
The set $u^{-1}(\{M\})$ is the preimage of the closed set $\{M\}$ under the continuous map u and so is closed. Therefore $S = u^{-1}(\{M\}) \cap \Omega$ is a closed subset of Ω.
We have shown that S is a nonempty open and closed subset of the connected set Ω. Therefore $S = \Omega$, which implies that $u = M =$ constant in Ω, as required. The weak maximum principle follows easily from this (see Q3).

(iii) Subharmonic functions do not satisfy the minimum principle

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

For example, take $\Omega = (-1, 1)$, $u : [-1, 1] \to \mathbb{R}$, $u(x) = x^2$. Then $-u''(x) = -2 < 0$ and so u is subharmonic. But the minimum value of u is 0, which is attained at $x = 0 \in \Omega$, not on the boundary of Ω.

3. *Strong maximum principle $\implies$ weak maximum principle.* By the strong maximum principle:
Either: u is constant, in which case it is trivial that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Or: u is not constant, in which case the strong maximum principle implies that, for all $\boldsymbol{x} \in \Omega$,

$$u(\boldsymbol{x}) < \max_{\overline{\Omega}} u,$$

i.e., the maximum of u over $\overline{\Omega}$ is not attained in Ω. Since $\overline{\Omega} = \Omega \cup \partial\Omega$, it follows that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

as required.

4. *The strong maximum principle is false if Ω is not connected.* Simply take $\Omega_1 = B_1((2,0))$, $\Omega_2 = B_1((-2,0))$, $\Omega = \Omega_1 \cup \Omega_2$, and define $u : \overline{\Omega} \to \mathbb{R}$ by

$$u(\boldsymbol{x}) = \begin{cases} 3 & \text{if } \boldsymbol{x} \in \overline{\Omega}_1, \\ 4 & \text{if } \boldsymbol{x} \in \overline{\Omega}_2. \end{cases}$$

Clearly, $u \in C^2(\Omega_1 \cup \Omega_2)$ and $u \in C(\overline{\Omega}_1 \cup \overline{\Omega}_2)$, while $\Omega_1 \cup \Omega_2$ is clearly disconnected. Finally, $\max_{\overline{\Omega}_1 \cup \overline{\Omega}_2} u = \max_{\overline{\Omega}_2} = u(-2,0)$, which is an interior point. Yet, the function is not constant.

5. *Minimum principles and an application: Positivity of solutions.*

(i) First we state the minimum principles: Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. Let $u : \overline{\Omega} \to \mathbb{R}$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω.

(a) Weak minimum principle: u attains its minimum on the boundary of Ω, i.e.,

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

(b) Strong minimum principle: If u attains its minimum in the interior of Ω, then u is constant, i.e., if there exists $\boldsymbol{x}_0 \in \Omega$ such that

$$u(\boldsymbol{x}_0) = \min_{\overline{\Omega}} u$$

then u is constant in Ω.

These can be proved as follows:

(a) Weak minimum principle: Let $\tilde{u} = -u$. Then $\tilde{u}$ is harmonic since u is harmonic. Therefore by the weak maximum principle

$$\begin{aligned} \min_{\overline{\Omega}} u &= -\max_{\overline{\Omega}}(-u) \\ &= -\max_{\overline{\Omega}} \tilde{u} \\ &= -\max_{\partial\Omega} \tilde{u} \\ &= -\max_{\partial\Omega}(-u) \\ &= \min_{\partial\Omega} u \end{aligned}$$

as required.

(b) Strong minimum principle: If u attains its minimum at $\boldsymbol{x}_0 \in \Omega$, then the harmonic function $\tilde{u} = -u$ attains its maximum at $\boldsymbol{x}_0$. By the strong maximum principle, $\tilde{u}$ is constant. Therefore u is constant.

(ii) Since u is harmonic it satisfies the strong minimum principle. Therefore:

Either: u is constant, in which case for all $\boldsymbol{x} \in \Omega$

$$u(\boldsymbol{x}) = u(\boldsymbol{x}_0) = g(\boldsymbol{x}_0) > 0,$$

as the function is continuous up to the boundary.

Or: u is not constant, in which case the strong minimum principle implies that, for all $\boldsymbol{x} \in \Omega$,

$$u(\boldsymbol{x}) > \min_{\boldsymbol{y} \in \partial\Omega} u(\boldsymbol{y}) = \min_{\boldsymbol{y} \in \partial\Omega} g(\boldsymbol{y}) \geq 0.$$

In either case $u(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \Omega$, as required.

6. *Another application of the maximum principle: Bounds on solutions.*

(a) Let $\Omega = (-1,1) \times (-1,1)$. We have

$$\min_{\partial\Omega} u = \min_{(x,y)\in\partial\Omega} (x^2+y^2) = 1,$$
$$\max_{\partial\Omega} u = \max_{(x,y)\in\partial\Omega} (x^2+y^2) = 2.$$

We know that u is not constant (since $u(x,y) = x^2+y^2$ on $\partial\Omega$). Therefore the strong maximum principle for harmonic functions implies that

$$\min_{\partial\Omega} u < u(0,0) < \max_{\partial\Omega} u \quad\Longleftrightarrow\quad 1 < u(0,0) < 2.$$

(b) Let

$$v = \frac{47}{40} - \frac{1}{5}(x^4 - 6x^2y^2 + y^4).$$

Then

$$v_x = -\frac{4}{5}x^3 + \frac{12}{5}xy^2, \qquad v_{xx} = -\frac{12}{5}x^2 + \frac{12}{5}y^2,$$
$$v_y = -\frac{4}{5}y^3 + \frac{12}{5}x^2y, \qquad v_{yy} = -\frac{12}{5}y^2 + \frac{12}{5}x^2,$$

and therefore v is harmonic.
Let $\Gamma_1 = \{(x,y) : |x| < 1, |y| = 1\}$. If $(x,y) \in \Gamma_1$, then

$$v(x,y) - 1 - x^2 = \frac{7}{40} - \frac{1}{5}x^4 + \frac{6}{5}x^2 - \frac{1}{5} - x^2 = -\frac{1}{40} - \frac{1}{5}x^4 + \frac{1}{5}x^2 =: f(x).$$

Let's find the infimum and supremum of f on Γ_1. We have

$$f'(x) = 0 \quad\Longleftrightarrow\quad -\frac{4}{5}x^3 + \frac{2}{5}x = 0 \quad\Longleftrightarrow\quad \frac{2}{5}x\left(-2x^2+1\right) = 0.$$

Therefore the critical points of f are

$$0, \quad \pm\frac{1}{\sqrt{2}}.$$

The maximum and minimum points of f on $\overline{\Gamma_1}$ are attained at the critical points of f or at the end points $x = \pm 1$. We have

$$f(\pm 1) = -\frac{1}{40} = -0.025, \quad f(0) = -\frac{1}{40} = -0.025, \quad f\left(\pm\frac{1}{\sqrt{2}}\right) = \frac{1}{40} = 0.025.$$

Therefore

$$-0.025 \le v(x,y) - 1 - x^2 \le 0.025 \quad \text{for all } (x,y) \in \Gamma_1$$

as required.
Let $\Gamma_2 = \{(x,y) : |x| = 1, |y| < 1\}$. By symmetry of v in x and y,

$$-0.025 \le v(x,y) - 1 - y^2 \le 0.025 \quad \text{for all } (x,y) \in \Gamma_2.$$

Let $w = v - u$. Then w is harmonic and $w = v - x^2 - y^2$ on $\partial\Omega$. We can decompose $\partial\Omega$ as

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \{(-1,-1), (-1,1), (1,-1), (1,1)\}.$$

For $(x,y) \in \partial\Omega$,

$$w(x,y) = \begin{cases} v - 1 - x^2 & \text{if } (x,y) \in \Gamma_1, \\ v - 1 - y^2 & \text{if } (x,y) \in \Gamma_2, \\ -\frac{1}{40} & \text{if } |x| = |y| = 1. \end{cases}$$

Clearly w is not constant. Therefore by the strong maximum principle

$$-\frac{1}{40} = \min_{\partial\Omega} w < w(0,0) < \max_{\partial\Omega} w = \frac{1}{40}.$$

Therefore

$$-\frac{1}{40} < \underbrace{v(0,0)}_{=\frac{47}{40}} - u(0,0) < \frac{1}{40} \iff \underbrace{\frac{46}{40}}_{1.15} < u(0,0) < \underbrace{\frac{48}{40}}_{1.2}$$

as desired. We have an estimate of $u(0,0)$ correct to two significant figures!

7. *Application of the maximum principle for subharmonic functions: Comparison theorems.* Let $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} -\Delta v &= f_1 - f_2 \quad \text{in } \Omega, \\ v &= g_1 - g_2 \quad \text{on } \partial\Omega. \end{aligned}$$

By assumption, $f_1 - f_2 \le 0$ and so v is subharmonic. Therefore it satisfies the maximum principle

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v = \max_{\partial\Omega}(g_1 - g_2) \le 0.$$

Therefore $v \le 0$ and $u_1 \le u_2$, as required.

8. *Maximum principles for more general elliptic problems.*

(i) Consider the one-dimensional steady convection-diffusion equation

$$-\alpha u'' + \beta u' = 0 \quad \text{in } (a,b)$$

where α and β are constants, $\alpha > 0$. Let $v = u'$. Then

$$-\alpha v' + \beta v = 0 \quad \Longrightarrow \quad \left(e^{-\frac{\beta}{\alpha}x} v\right)' = 0.$$

Therefore

$$v(x) = c e^{\frac{\beta}{\alpha}x}$$

for some constant c. Hence

$$u(x) = \frac{c\alpha}{\beta} e^{\frac{\beta}{\alpha}x} + d$$

for some constant d. If $c = 0$, then u is constant. Otherwise

$$u'(x) = v(x) = c e^{\frac{\beta}{\alpha}x}$$

and so u is strictly increasing if $c > 0$ and strictly decreasing if $c < 0$. Therefore u attains its maximum and minimum on the boundary of (a,b).

(ii) Let u satisfy Poisson's equation

$$-u'' = f \quad \text{in } (a,b)$$

where f is a constant. Then u is a quadratic polynomial. It is easy to see that if $f < 0$, then u satisfies a weak maximum principle, and if $f > 0$, then u satisfies a weak minimum principle. See Q2 for the two-dimensional case.

(iii) Consider the equation

$$-u'' + cu = 0 \quad \text{in } \Omega$$

with $c > 0$, $\Omega = (a, b)$. The solution has the form

$$u(x) = A\exp(\sqrt{c}x) + B\exp(-\sqrt{c}x)$$

where A and B are constants. We can assume that $A \neq 0$ and $B \neq 0$, otherwise the result is obvious. If $B/A < 0$, then u is either increasing (if $A > 0$, $B < 0$) or decreasing (if $A < 0$, $B > 0$) and hence u attains its maximum and minimum on the boundary of Ω. It follows that $\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$. If $B/A > 0$, then u has a unique critical point:

$$u'(x_0) = 0 \quad \Longleftrightarrow \quad \exp(2\sqrt{c}\,x_0) = \frac{B}{A} \quad \Longleftrightarrow \quad x_0 = \frac{1}{\sqrt{c}}\ln\left(\frac{B}{A}\right)$$

and the critical value of u is

$$u(x_0) = A\left(\frac{B}{A}\right)^{1/2} + B\left(\frac{B}{A}\right)^{-1/2}.$$

If $x_0 \notin \Omega$, then u is increasing or decreasing on Ω and so $\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|$ as before. Assume that $x_0 \in \Omega$. We consider two case: $A, B > 0$ and $A, B < 0$.
If $A, B > 0$, then $u(x) > 0$ for all $x \in (a, b)$ and $u''(x_0) = cu(x_0) > 0$, which implies that x_0 is a local minimum point of u. Therefore $u = |u|$ attains its maximum on the boundary of Ω, as required.
If $A, B < 0$, then $u(x) < 0$ for all $x \in (a, b)$ and $u''(x_0) = cu(x_0) < 0$, which implies that u_0 is a local maximum point of u. Therefore

$$\max_{\overline{\Omega}} |u| = -\min_{\overline{\Omega}} u = -\min_{\partial\Omega} u = \max_{\partial\Omega} |u|$$

as required.
If $c < 0$, then the maximum principle does not hold since

$$u(x) = A\sin(\sqrt{-c}x) + B\cos(\sqrt{-c}x)$$

for some constants A and B. For example, take $a = 0$, $b = 2\pi$, $c = -1$, $A = 1$, $B = 0$. Then

$$\max_{\overline{\Omega}} |u| = 1, \qquad \max_{\partial\Omega} |u| = 0.$$

9. *Maximum principles for 4th-order elliptic PDEs?* In general, 4th-order elliptic PDEs do not satisfy a maximum principle. For example, if $u'''' = 0$ on (a, b), then u is a cubic polynomial, which need not attain is maximum or minimum on the boundary of (a, b). If $-u'''' = f$ on (a, b), where $f < 0$ is a constant, then u is a quartic polynomial, which again need not attain its maximum on the boundary of (a, b).

10. *Regularity Theorem: Harmonic functions are C^∞.*

(i) Observe that $\eta = 0$ outside the disc $B_1(\mathbf{0})$. Therefore $\operatorname{supp}(\eta) = \overline{B_1(\mathbf{0})}$ and so $\operatorname{supp}(\eta_\varepsilon) = \overline{B_\varepsilon(\mathbf{0})}$. For the rest of the problem it is convenient to write $\eta(\boldsymbol{x}) = \phi(|\boldsymbol{x}|)$ where $\phi : [0, \infty) \to \mathbb{R}$ is defined by

$$\phi(r) = \begin{cases} C\exp\left(-\frac{1}{1-r^2}\right) & \text{if } r < 1, \\ 0 & \text{if } r \geq 1. \end{cases}$$

Then $\eta_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2}\phi(\frac{|\boldsymbol{x}|}{\varepsilon})$. Observe that

$$\int_{B_1(\mathbf{0})} \phi(|\boldsymbol{x}|)\, d\boldsymbol{x} = \int_{B_1(\mathbf{0})} \eta(\boldsymbol{x})\, d\boldsymbol{x} = C \int_{B_1(\mathbf{0})} e^{-\frac{1}{1-|\boldsymbol{x}|^2}}\, d\boldsymbol{x} = 1$$

by definition of C. We compute

$$\begin{aligned}
\int_{\mathbb{R}^2} \eta_\varepsilon(\boldsymbol{x})\, d\boldsymbol{x} &= \int_{B_\varepsilon(\mathbf{0})} \eta_\varepsilon(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \int_0^{2\pi}\int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) r\, dr d\theta && \text{(polar coordiates)} \\
&= \int_0^{2\pi}\int_0^1 \frac{1}{\varepsilon^2}\phi(s)\, s\varepsilon\, \varepsilon ds d\theta && \text{(change of variables: } s = \frac{r}{\varepsilon}) \\
&= \int_0^{2\pi}\int_0^1 \phi(s) s\, ds d\theta \\
&= \int_{B_1(\mathbf{0})} \phi(|\boldsymbol{x}|)\, d\boldsymbol{x} && \text{(back to Cartesian coordinates)} \\
&= 1.
\end{aligned}$$

(ii) Take $\boldsymbol{x} \in \Omega_\varepsilon$. Then

$$\begin{aligned}
u_\varepsilon(\boldsymbol{x}) &= \int_{B_\varepsilon(\boldsymbol{x})} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) u(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \int_{B_\varepsilon(\boldsymbol{x})} \frac{1}{\varepsilon^2}\phi\left(\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\varepsilon}\right) u(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \int_0^{2\pi}\int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) u(\boldsymbol{x} + r(\cos\theta, \sin\theta))\, r\, dr d\theta && (\boldsymbol{y} = \boldsymbol{x} + r(\cos\theta, \sin\theta)) \\
&= \int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) \left(\int_0^{2\pi} u(\boldsymbol{x} + r(\cos\theta, \sin\theta))\, r\, d\theta\right) dr \\
&= \int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) \underbrace{\left(\int_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dL(\boldsymbol{y})\right)}_{=|\partial B_r(\boldsymbol{x})|\, u(\boldsymbol{x})} dr && \text{(mean-value formula)} \\
&= \int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) 2\pi r\, u(\boldsymbol{x})\, dr \\
&= u(\boldsymbol{x})\, 2\pi \int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) r\, dr \\
&= u(\boldsymbol{x}) \int_0^{2\pi}\int_0^\varepsilon \frac{1}{\varepsilon^2}\phi\left(\frac{r}{\varepsilon}\right) r\, dr d\theta \\
&= u(\boldsymbol{x}) \int_{B_\varepsilon(\mathbf{0})} \frac{1}{\varepsilon^2}\phi\left(\tfrac{|\boldsymbol{y}|}{\varepsilon}\right)\, d\boldsymbol{y} \\
&= u(\boldsymbol{x}) \int_{B_\varepsilon(\mathbf{0})} \eta_\varepsilon(\boldsymbol{y})\, d\boldsymbol{y} \\
&= u(\boldsymbol{x}).
\end{aligned}$$

11. $C^\infty \not\Rightarrow$ *analytic.* Consider the function $\eta : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then η is infinitely differentiable but it is not analytic since it does not have a convergent Taylor series expansion about the point $x = 1$:

$$\sum_{k=0}^{\infty} \frac{\eta^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} \frac{0}{k!}(x-1)^k = 0,$$

but η is nonzero in any neighbourhood of $x = 1$. In general, nonzero analytic functions cannot have compact support.

12. *Non-negative harmonic functions on* $\mathbb{R}^n$ *are constant.*

(i) We have

$$\begin{aligned} u(\boldsymbol{x}) &= \fint_{B_r(\boldsymbol{x})} u(\boldsymbol{z})\, d\boldsymbol{z} && \text{(mean-value formula)} \\ &= \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} \frac{1}{|B_R(\boldsymbol{y})|} \int_{B_r(\boldsymbol{x})} u(\boldsymbol{z})\, d\boldsymbol{z} \\ &\leq \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} \frac{1}{|B_R(\boldsymbol{y})|} \int_{B_R(\boldsymbol{y})} u(\boldsymbol{z})\, d\boldsymbol{z} && \text{(since } u > 0 \text{ and } B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})) \\ &= \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} u(\boldsymbol{y}) && \text{(mean-value formula)} \end{aligned}$$

as required.

(ii) Let $\boldsymbol{z} \in B_r(\boldsymbol{x})$. Then

$$|\boldsymbol{z} - \boldsymbol{y}| = |\boldsymbol{z} - \boldsymbol{x} + \boldsymbol{x} - \boldsymbol{y}| \leq |\boldsymbol{z} - \boldsymbol{x}| + |\boldsymbol{x} - \boldsymbol{y}| < r + |\boldsymbol{x} - \boldsymbol{y}| = R.$$

Therefore $\boldsymbol{z} \in B_R(\boldsymbol{y})$ and hence $B_r(\boldsymbol{x}) \subset B_R(\boldsymbol{y})$. We have

$$\frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} = \frac{R^n \alpha(n)}{r^n \alpha(n)} = \frac{R^n}{(R - |\boldsymbol{x} - \boldsymbol{y}|)^n} = \frac{1}{1 - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{R}} \to 1 \quad \text{as } R \to \infty.$$

(iii) If $r = R - |\boldsymbol{x} - \boldsymbol{y}|$, then by parts (i) and (ii),

$$u(\boldsymbol{x}) \leq \frac{|B_R(\boldsymbol{y})|}{|B_r(\boldsymbol{x})|} u(\boldsymbol{y}) \to u(\boldsymbol{y}) \quad \text{as } R \to \infty.$$

Therefore

$$u(\boldsymbol{x}) \leq u(\boldsymbol{y}).$$

Interchanging the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ gives

$$u(\boldsymbol{y}) \leq u(\boldsymbol{x}).$$

Therefore $u(\boldsymbol{x}) = u(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and hence u is a constant function.

13. *Proof of Liouville's Theorem.* Since u is bounded, then there exists $M > 0$ such that $u(\boldsymbol{x}) > -M$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Therefore the harmonic function $v = u + M > 0$ on $\mathbb{R}^n$. But positive harmonic functions on $\mathbb{R}^n$ are constant by Q12. Therefore v, and hence u, are constant.

14. *An application of Liouville's Theorem: 'Uniqueness' for Poisson's equation in* $\mathbb{R}^3$. Let $u_1 = \Phi * f$ where Φ is the fundamental solution of Poisson's equation in $\mathbb{R}^n$ with $n = 3$:

$$\Phi(\boldsymbol{x}) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\boldsymbol{x}|^{n-2}} = \frac{1}{4\pi} \frac{1}{|\boldsymbol{x}|}.$$

(Recall that $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and hence $\alpha(3) = \frac{4}{3}\pi$.) Let u_2 be any bounded solution of Poisson's equation in $\mathbb{R}^3$. Then $w = u_2 - u_1$ is a harmonic function since $-\Delta u_1 = f$ and $-\Delta u_2 = f$ in $\mathbb{R}^3$. We show that u_1 is bounded: Since $f \in C_c^2(\mathbb{R}^3)$ has compact support, there exists $R > 0$ such that $\operatorname{supp}(f) \subset B_R(\boldsymbol{0})$. In particular, $f = 0$ in $\mathbb{R}^3 \setminus B_R(\boldsymbol{0})$. Therefore

$$\begin{aligned} |u_1(\boldsymbol{x})| &= |(\Phi * f)(\boldsymbol{x})| \\ &= \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} \right| \\ &\le \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} \\ &= \frac{1}{4\pi} \int_{B_R(\boldsymbol{0})} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} \\ &\le \frac{1}{4\pi} \max_{B_R(\boldsymbol{0})} |f| \int_{B_R(\boldsymbol{0})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y}. \end{aligned}$$

We just need to show that

$$\int_{B_R(\boldsymbol{0})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y}$$

is uniformly bounded in $\boldsymbol{x}$. This is more fiddly than you would expect. We consider two cases: $|\boldsymbol{x}| \le 2R$ and $|\boldsymbol{x}| > 2R$.

If $|\boldsymbol{x}| \le 2R$, then $B_R(\boldsymbol{0}) \subset B_{3R}(\boldsymbol{x})$ (draw a sketch to convince yourself of this) and so

$$\begin{aligned} \int_{B_R(\boldsymbol{0})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} &< \int_{B_{3R}(\boldsymbol{x})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} \\ &= \int_{B_{3R}(\boldsymbol{0})} \frac{1}{|\boldsymbol{z}|} \, d\boldsymbol{z} && (\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}) \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_0^{3R} \frac{1}{r} r^2 \sin\theta \, dr d\theta d\phi && \text{(spherical polar coordinates)} \\ &= 2\pi \left. \frac{1}{2} r^2 \right|_{r=0}^{3R} \left. (-\cos\theta) \right|_{\theta=0}^{\pi} \\ &= 18\pi R^2. \end{aligned}$$

If $|\boldsymbol{x}| > 2R$, then for all $\boldsymbol{y} \in B_R(\boldsymbol{0})$

$$|\boldsymbol{x}-\boldsymbol{y}| > R \quad \implies \quad \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} < \frac{1}{R}$$

and so

$$\int_{B_R(\boldsymbol{0})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} < \int_{B_R(\boldsymbol{0})} \frac{1}{R} \, d\boldsymbol{y} = \frac{1}{R} |B_R(\boldsymbol{0})| = \frac{1}{R} \frac{4}{3} \pi R^3 = \frac{4}{3} \pi R^2 < 18\pi R^2.$$

Therefore, for all $\boldsymbol{x} \in \mathbb{R}^3$,

$$|u_1(\boldsymbol{x})| \le \frac{1}{4\pi} \max_{B_R(\boldsymbol{0})} |f| \int_{B_R(\boldsymbol{0})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} < \frac{1}{4\pi} \max_{B_R(\boldsymbol{0})} |f| \, 18\pi R^2 = \frac{18}{4} R^2 \max_{B_R(\boldsymbol{0})} |f|.$$

Hence u_1 is bounded. Since u_1 and u_2 are bounded, then w is a bounded harmonic function on $\mathbb{R}^3$. By Liouville's Theorem $w = c =$ constant. Therefore

$$u_2 - u_1 = c \quad \iff \quad u_2 = u_1 + c = \Phi * f + c$$

as required.

This argument can be extended to $\mathbb{R}^n$ for any $n \geq 3$. It does not work for $n = 2$ since $u_1 = \Phi * f$ is not necessarily bounded in $\mathbb{R}^2$ since $\Phi(\boldsymbol{x}) = -\frac{1}{2\pi}\log|\boldsymbol{x}|$ blows up as $|\boldsymbol{x}| \to \infty$. For the case $n \geq 3$, $\Phi(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$, and it converges to 0 sufficiently fast in order for $\Phi * f$ to be bounded.

15. *An obstacle to uniqueness for Laplace's equation: Unbounded domains.*

 (i) We can build a nontrivial solution using the fundamental solution of Poisson's equation:

$$u(\boldsymbol{x}) = \begin{cases} \log|\boldsymbol{x}| & \text{if } n = 2, \\ |\boldsymbol{x}|^{2-n} - 1 & \text{if } n \geq 3. \end{cases}$$

 (ii) Simply take $u(\boldsymbol{x}) = x_n$.

16. *Eigenvalues of the negative Laplacian.* By Exercise Sheet 4, Q11, the eigenvalues are positive, $\lambda > 0$. Therefore we can write each eigenvalue as $\lambda = \omega^2$ for some $\omega \in (0, \infty)$. Then

$$-u''(x) = \omega^2 u(x), \quad x \in (0, 2\pi).$$

Recall from ODE theory (see page 21 of the lecture notes) that solutions of this ODE have the form

$$u(x) = A\cos(\omega x) + B\sin(\omega x)$$

for some constants $A, B \in \mathbb{R}$. The boundary condition $u(0) = 0$ implies that $A = 0$. The boundary condition $u(2\pi) = 0$ gives

$$B\sin(2\pi\omega) = 0.$$

Since $u \neq 0$, then $B \neq 0$. Since $\omega > 0$, it follows that

$$2\pi\omega \in \{n\pi : n \in \mathbb{N}\}.$$

Therefore $\omega = \frac{n}{2}$ and the eigenfunction-eigenvalue pairs are

$$(u_n(x), \lambda_n) = \left(B\sin\left(\frac{nx}{2}\right), \frac{n^2}{4}\right), \quad n \in \mathbb{N}, \quad B \in \mathbb{R}.$$

In particular, there are countably many eigenvalues.

17. *Connection between holomorphic functions and harmonic functions.*

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic (complex analytic) function with real and imaginary parts u and v:

$$f(x + iy) = u(x, y) + iv(x, y).$$

The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x.$$

Therefore

$$\Delta u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

and

$$\Delta v = v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0.$$

Completing the table gives

Harmonic Functions	Holomorphic Functions
Mean-Value Formula	Cauchy Integral Formula
Maximum Principle	Maximum Modulus Principle
Liouville's Theorem	Liouville's Theorem

See Remark 5.3 in the lecture notes for an explanation of why the Cauchy integral formula implies the mean-value formula.