Partial Differential Equations III/IV
Exercise Sheet 6

1. *The Fourier transform: The heat equation with source term.*

   (i) Verify that
   $$x(t) = Ge^{\lambda t} + \int_0^t e^{\lambda(t-s)} F(s)\, ds$$
   satisfies the ODE
   $$\dot{x}(t) = \lambda x(t) + F(t), \qquad x(0) = G.$$
   This is an example of Duhamel's principle, which is a method for obtaining a solution of an inhomogeneous differential equation, in this case $\dot{x} - \lambda x = F$, from the corresponding homogeneous differential equation, in this case $\dot{x} - \lambda x = 0$.

   (ii) Consider the heat equation on $\mathbb{R}$ with source term $f(x,t)$:
   $$\begin{aligned} u_t - ku_{xx} &= f \quad \text{in } \mathbb{R} \times (0,\infty), \\ u &= g \quad \text{for } t = 0. \end{aligned}$$
   Use the Fourier transform and part (i) to derive the solution
   $$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s) f(y,s)\, dy\, ds$$
   where Φ is the fundamental solution of the heat equation in $\mathbb{R}$.

2. *The Fourier transform: The transport equation.*

   (i) Let $v \in L^1(\mathbb{R})$ and define $\tau_a v \in L^1(\mathbb{R})$ by $\tau_a v(x) = v(x-a)$, which is the translation of v by $a \in \mathbb{R}$. Use a change of variables to prove that
   $$\widehat{\tau_a v}(\xi) = e^{-i\xi a} \hat{v}(\xi).$$

   (ii) Use the Fourier transform and part (i) to derive the solution $u(x,t) = g(x-ct)$ of the transport equation
   $$u_t + cu_x = 0 \text{ for } (x,t) \in \mathbb{R} \times (0,\infty), \quad u(x,0) = g(x) \text{ for } x \in \mathbb{R}.$$

3. *The Fourier transform: Schrödinger's equation.* Consider Schrödinger's equation
   $$\begin{aligned} iu_t + u_{xx} &= 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{1}$$
   where u and g are complex-valued.

   (i) Use the Fourier transform to derive the solution
   $$u(x,t) = \frac{1}{\sqrt{4\pi i t}} \int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{4t}} g(y)\, dy.$$

(ii) Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{C}$ satisfy (1). We write $u(\cdot, t)$ to denote the function $x \mapsto u(x, t)$ for fixed t. Assume that $g, u(\cdot, t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and that $u_t(\cdot, t), u_{xx}(\cdot, t) \in L^1(\mathbb{R})$ for all $t > 0$. Use the Fourier transform to prove that

$$\|u(\cdot, t)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} \quad \forall\, t > 0.$$

This can also be proved using the energy method.
Hint: Use the fact that the Fourier transform preserves the L^2–norm: $\|\hat{g}\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}$ and $\|\hat{u}(\cdot, t)\|_{L^2(\mathbb{R})} = \|u(\cdot, t)\|_{L^2(\mathbb{R})}$ (you do not need to prove this).
Remark: This shows that the energy $E(t) := \|u(\cdot, t)\|^2_{L^2(\mathbb{R})}$ is constant. Recall that if u satisfies the heat equation on $\mathbb{R}$ with diffusion constant k, then the energy decays:

$$\frac{d}{dt} E(t) = \frac{d}{dt} \|u(\cdot, t)\|^2_{L^2(\mathbb{R})} = -2k \|u_x(\cdot, t)\|^2_{L^2(\mathbb{R})} \leq 0.$$

Schrödinger's equation is an example of a *dispersive equation*, where energy is conserved, whereas the heat equation is an example of a *diffusion* equation, where energy decays.

4. *The Fourier transform: The wave equation.* Use the Fourier transform to derive the solution

$$u(x, t) = \frac{1}{2}[g(x - ct) + g(x + ct)]$$

of the wave equation

$$\begin{aligned} u_{tt} = c^2 u_{xx} \quad & \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = g(x) \quad & \text{for } x \in \mathbb{R}, \\ u_t(x, 0) = 0 \quad & \text{for } x \in \mathbb{R}, \end{aligned}$$

where the constant $c > 0$ is the wave speed. This is known as *D'Alembert's solution.*
Hint: Use Q2(i) and the fact that $\cos(c\xi t) = [\exp(ic\xi t) + \exp(-ic\xi t)]/2$.

5. *The Fourier transform of a derivative.* Let $u, u' \in L^1(\mathbb{R})$. Use integration by parts to prove that

$$\widehat{u'}(\xi) = i\xi \hat{u}(\xi).$$

6. *The Fourier transform of a convolution.* Let $u, v \in L^1(\mathbb{R})$. Prove that

$$\widehat{u * v} = \sqrt{2\pi}\, \hat{u}\hat{v}.$$

Hint: By definition

$$\widehat{u * v}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u * v)(x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(z) v(x - z)\, dz \right) e^{-i\xi x}\, dx.$$

The trick is to write

$$e^{-i\xi x} = e^{-i\xi z} e^{-i\xi(x - z)}$$

and then to interchange the order of integration.

7. *Proof of the Sobolev embedding using the Fourier transform.* In this question we use the Fourier transform to give an alternative proof of the Sobolev embedding $\|u\|_{L^\infty(I)} \leq C\|u\|_{H^1(I)}$ for the case $I = \mathbb{R}$. Assume that $u \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\hat{u} \in L^1(\mathbb{R})$, and $u' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Recall that

$$\|u\|_{H^1(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + \|u'\|^2_{L^2(\mathbb{R})} \right)^{1/2}.$$

(i) Prove that

$$\|u\|_{H^1(\mathbb{R})}^2 = \int_{-\infty}^{\infty} (1+|\xi|^2)|\hat{u}(\xi)|^2\, d\xi.$$

Hint: Use the fact that the Fourier transform preserves the L^2–norm: $\|\hat{v}\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})}$ for all $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (you do not need to prove this).

(ii) Prove that there exists a constant $C > 0$ such that

$$\|\hat{u}\|_{L^1(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}.$$

Hint: Write

$$\|\hat{u}\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |\hat{u}(\xi)| d\xi = \int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)^{1/2}} (1+|\xi|^2)^{1/2}\, |\hat{u}(\xi)|\, d\xi.$$

(iii) Use the Fourier Inversion Theorem to prove that

$$\|u\|_{L^\infty(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}.$$

8. *Fundamental Solution of the Heat Equation.* The Fundamental Solution of the Heat Equation in $\mathbb{R}^n$ is

$$\Phi(\boldsymbol{x},t) = \frac{1}{(4\pi k t)^{\frac{n}{2}}}\, e^{-\frac{|\boldsymbol{x}|^2}{4kt}}, \quad \boldsymbol{x} \in \mathbb{R}^n,\ t > 0.$$

Verify that Φ satisfies the heat equation

$$\Phi_t(\boldsymbol{x},t) = k\Delta\Phi(\boldsymbol{x},t)$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, $t > 0$.
Remark: It can be shown that $\Phi \to \delta$ as $t \to 0$ in the sense of distributions.

9. *Finite speed of propagation for a degenerate diffusion equation.* Define $\Phi : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ by

$$\Phi(x,t) = \max\left\{ \frac{1}{2}\left(\frac{3}{k\pi t}\right)^{\frac{1}{3}} - \frac{1}{6k}\frac{x^2}{t},\ 0 \right\}.$$

Let $a(\Phi) = k\Phi$, where $k > 0$ is a constant.

(i) Show that Φ satisfies the degenerate diffusion equation

$$\Phi_t = (a(\Phi)\Phi_x)_x$$

for all $x \in \mathbb{R}$, $t > 0$, except for

$$|x| = 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$$

where it is not differentiable. (It can also be shown that Φ satisfies the the degenerate diffusion equation in all of $\mathbb{R} \times (0,\infty)$ in a suitable weak sense.)

(ii) Show that the map $x \mapsto \Phi(x,t)$ has compact support for all $t > 0$. Therefore, unlike the heat/diffusion equation, the *degenerate* diffusion equation has *finite* speed of propagation.

Remark: Observe that the diffusion coefficient a vanishes when $\Phi = 0$. Compare this to the case of the heat equation, where $a = k > 0$ is strictly positive. For the 4H students: Just like for the Fundamental Solution of the Heat Equation, it can be shown that $\Phi \to \delta$ as $t \to 0$ in the sense of distributions.

10. *The mathematical equation that caused the banks to crash.* The Black-Scholes PDE, or "the mathematical equation that caused the banks to crash" (Ian Stewart, The Observer, 12 Feb 2012), is the parabolic PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $V(S,t)$ is the price of a European option as a function of the stock price S at time t, r is the risk-free interest rate, and σ is the volatility of the stock (see Wikipedia `https://en.wikipedia.org/wiki/Black-Scholes_equation`). Consider the change of variables

$$\tau = T - t, \qquad x = \ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau, \qquad u(x,\tau) = Ce^{r\tau}V(S(x,\tau), t(x,\tau)),$$

where T, C, K are constants. Show that the Black-Scholes PDE is the heat equation in disguise:

$$u_\tau = \frac{1}{2}\sigma^2 u_{xx}.$$

So the heat equation is "the mathematical equation that caused the banks to crash"!
Remark: You can read Ian Stewart's article here: https://www.theguardian.com/science/2012/feb/12/black-scholes-equation-credit-crunch

11. *The energy method: Uniqueness for the heat equation in a time dependent domain.* Let $k > 0$, $T > 0$ be constants. Let $a, b : [0,T] \to \mathbb{R}$ be smooth functions with $a(t) < b(t)$ for all t. Let $U \subset \mathbb{R} \times (0,T]$ be the non-cylindrical domain
$$U = \{(x,t) \in \mathbb{R} \times (0,T] \,:\, a(t) < x < b(t)\}.$$
Consider the heat equation

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } (x,t) \in U, \\ u(a(t),t) &= g_1(t) \quad \text{for } t \in [0,T], \\ u(b(t),t) &= g_2(t) \quad \text{for } t \in [0,T], \\ u(x,0) &= u_0(x) \quad \text{for } x \in (a(0),b(0)). \end{aligned}$$

Use the energy method to prove that this equation has at most one smooth solution.

12. *The energy method: Uniqueness for a 4th-order heat equation.* Let $k > 0$, $T > 0$. Prove that there exists at most one smooth solution $u : [a,b] \times [0,T] \to \mathbb{R}$ of the 4th-order heat equation

$$\begin{aligned} u_t + ku_{xxxx} &= f \quad \text{for } (x,t) \in (a,b) \times (0,T], \\ u(a,t) = u(b,t) &= 0 \quad \text{for } t \in [0,T], \\ u_x(a,t) = u_x(b,t) &= 0 \quad \text{for } t \in [0,T], \\ u(x,0) &= u_0(x) \quad \text{for } x \in (a,b). \end{aligned}$$

Since the equation is 4th-order, we prescribe boundary conditions on both u and u_x. Why do we consider $u_t + ku_{xxxx} = 0$ to be the 4th-order version of the heat equation instead of $u_t - ku_{xxxx} = 0$, which at first sight seems to be closer to the heat equation $u_t - ku_{xx} = 0$?

13. *Asymptotic behaviour of the heat equation with time independent data.* Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with smooth boundary. Let $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be a smooth function satisfying

$$\begin{aligned} u_t(\boldsymbol{x},t) - k\Delta u(\boldsymbol{x},t) &= f(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ u(\boldsymbol{x},t) &= g(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x},t) \in \partial\Omega \times [0,\infty), \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega, \end{aligned}$$

where f, g, u_0 are given smooth functions. Let $v : \overline{\Omega} \to \mathbb{R}$ be a smooth, time independent solution of the same equation:

$$\begin{aligned} -k\Delta v(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega, \\ v(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \partial\Omega. \end{aligned}$$

Define $w(\boldsymbol{x}, t) = u(\boldsymbol{x}, t) - v(\boldsymbol{x})$. Use the energy method to prove that $w \to 0$ in $L^2(\Omega)$ as $t \to \infty$. In other words, if the source term f and boundary data g are independent of time, then the solution u of the heat equation converges to the solution v of Poisson's equation in the L^2–norm as $t \to \infty$.

14. *Asymptotic behaviour of the heat equation with time independent data in the L^∞–norm.* Let $k > 0$ be a constant and let $u : [a, b] \times [0, \infty) \to \mathbb{R}$ be a smooth function satisfying the heat equation

$$\begin{aligned} u_t(x,t) - ku_{xx}(x,t) &= f(x) \quad \text{for } (x,t) \in (a,b) \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \text{for } x \in (a,b), \\ u(a,t) = u(b,t) &= 0 \quad \text{for } t \in [0,\infty), \end{aligned}$$

where u_0 and f are smooth functions. Let $v : [a, b] \to \mathbb{R}$ be the unique solution of

$$\begin{aligned} -kv_{xx}(x) &= f(x) \quad \text{for } x \in (a,b), \\ v(a) &= v(b) = 0. \end{aligned}$$

Define $w(x, t) = u(x, t) - v(x)$.

(i) Prove that w satisfies

$$\frac{d}{dt}\int_a^b w^2(x,t)\,dx = -2k\int_a^b w_x^2(x,t)\,dx.$$

(ii) Prove that $w \to 0$ in $L^2([a, b])$ as $t \to \infty$.

(iii) Prove that $w_t \to 0$ in $L^2([a, b])$ as $t \to \infty$.
Hint: Show that w_t satisfies a heat equation.

(iv) Prove that $w_x \to 0$ in $L^2([a, b])$ as $t \to \infty$.
Hint: By part (i),

$$\int_a^b w_x^2(x,t)\,dx = -\frac{1}{k}\int_a^b w(x,t)w_t(x,t)\,dx.$$

(v) Conclude that $w \to 0$ in $L^\infty([a, b])$ as $t \to \infty$.

15. *Applications of the maximum principle: Uniqueness and bounds on solutions.* This question appeared on the May 2012 exam, Q9(b),(c).
Given $T > 0$, let $\Omega := (a, b)$ with $a < b$ and let $\Omega_T := (a, b) \times (0, T]$.

(i) Show that in $C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ there exists at most one solution to the problem

$$u_t - u_{xx} = 1 \quad \text{on } \Omega_T$$

with $u = 0$ on the parabolic boundary $[a, b] \times \{0\} \cup \{a, b\} \times [0, T]$.

(ii) Assume that u is a solution to the problem in (i). Show that we have

$$0 \le u(x,t) \le t$$

for $(x, t) \in \Omega_T$.

16. *Application of the maximum principle: Comparison Principle.* For $i \in \{1, 2\}$, let u_i be a smooth function satisfying

$$\begin{aligned} \frac{\partial u_i}{\partial t}(\boldsymbol{x}, t) - k\Delta u_i(\boldsymbol{x}, t) &= f_i(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ u_i(\boldsymbol{x}, t) &= g_i(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x}, t) \in \partial\Omega \times [0, T], \\ u_i(\boldsymbol{x}, 0) &= u_i^0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega, \end{aligned}$$

where f_i, g_i, u_i^0 are given smooth functions. Assume that $f_2 \geq f_1$, $g_2 \geq g_1$, and $u_2^0 \geq u_1^0$. Prove that $u_2 \geq u_1$ in Ω_T.

17. *Eigenfunctions of the Laplacian and an application to the heat equation.* Let (λ_n, u_n), $n \in \mathbb{N}$, be eigenvalue-eigenfunction pairs for $-\Delta$ on Ω with zero Dirichlet boundary conditions, which means that $u_n \neq 0$ and that (λ_n, u_n) satisfies

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n \quad \text{in } \Omega, \\ u_n &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned}$$

In Exercise Sheet 4 we used the energy method to show that $\lambda_n \in \mathbb{R}$ and $\lambda_n > 0$ for all n. By relabelling if necessary, we can assume that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ (in fact it can be shown that $\lambda_1 < \lambda_2$). Let v satisfy the heat equation

$$\begin{aligned} v_t(\boldsymbol{x}, t) - k\Delta v(\boldsymbol{x}, t) &= 0 \quad \text{for } (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ v(\boldsymbol{x}, t) &= 0 \quad \text{for } (\boldsymbol{x}, t) \in \partial\Omega \times [0, \infty), \\ v(\boldsymbol{x}, 0) &= g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega. \end{aligned}$$

Roughly speaking, it can be shown that the set of eigenfunctions $\{u_n\}_{n\in\mathbb{N}}$ forms a basis for the vector space of smooth functions on Ω that vanish on $\partial\Omega$. By writing v and g with respect to this basis as

$$v(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} c_n(t) u_n(\boldsymbol{x}), \qquad g(\boldsymbol{x}) = \sum_{n=1}^{\infty} g_n u_n(\boldsymbol{x}),$$

show formally that

$$v(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} g_n e^{-k\lambda_n t} u_n(\boldsymbol{x}). \tag{2}$$

Remark: From expression (2) we see that the rate of convergence of v to 0 as $t \to \infty$ depends on the smallest eigenvalue of $-\Delta$. This should not come as a surprise: When we proved that $v \to 0$ as $t \to \infty$ using the energy method, we saw that the rate of convergence depends on the Poincaré constant (see Q13), and from Exercise Sheet 4, Q12 we know that the optimal Poincaré constant depends on the smallest eigenvalue of $-\Delta$.