Partial Differential Equations III/IV
Exercise Sheet 6: Solutions

1. *The Fourier transform: The heat equation with source term.*

   (i) By the Fundamental Theorem of Calculus

$$\begin{aligned}\dot{x}(t) &= \lambda G e^{\lambda t} + e^{\lambda(t-s)}F(s)\Big|_{s=t} + \int_0^t \lambda e^{\lambda(t-s)}F(s)\,ds \\ &= \lambda G e^{\lambda t} + F(t) + \lambda \int_0^t e^{\lambda(t-s)}F(s)\,ds \\ &= \lambda x(t) + F(t)\end{aligned}$$

   as claimed.

   (ii) Taking the Fourier transform of $u_t = ku_{xx} + f$ with respect to the x variable gives

$$\widehat{u_t} = k\widehat{u_{xx}} + \hat{f} \quad \Longleftrightarrow \quad \hat{u}_t(\xi,t) = k(i\xi)^2\hat{u}(\xi,t) + \hat{f}(\xi,t) = -k\xi^2\hat{u}(\xi,t) + \hat{f}(\xi,t).$$

   Taking the Fourier transform of the initial condition $u(x,0) = g(x)$ gives

$$\hat{u}(\xi,0) = \hat{g}(\xi).$$

   We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ:

$$\hat{u}_t = -k\xi^2\hat{u} + \hat{f}, \qquad \hat{u}(\xi,0) = \hat{g}(\xi).$$

   Applying part (i) with $x = \hat{u}$, $\lambda = -k\xi^2$, $F = \hat{f}$, $G = \hat{g}$ gives

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-k\xi^2 t} + \int_0^t e^{-k\xi^2(t-s)}\hat{f}(\xi,s)\,ds.$$

   Therefore

$$u(x,t) = \widecheck{\hat{g}(\xi)e^{-k\xi^2 t}} + \int_0^t \widecheck{e^{-k\xi^2(t-s)}\hat{f}(\xi,s)}\,ds. \tag{1}$$

   Recall that

$$\widehat{e^{-ax^2}}(\xi) = \frac{1}{\sqrt{2a}}\,e^{-\frac{\xi^2}{4a}}.$$

   Therefore

$$e^{-k\xi^2 t} = \sqrt{2a}\,\widehat{e^{-ax^2}}(\xi) \quad \text{for} \quad a = \frac{1}{4kt}.$$

   Since the product of Fourier transforms is the Fourier transform of a convolution, we obtain

$$\begin{aligned}\hat{g}(\xi)e^{-k\xi^2 t} &= \sqrt{2a}\,\hat{g}(\xi)\,\widehat{e^{-ax^2}}(\xi) \\ &= \sqrt{2a}\frac{1}{\sqrt{2\pi}}\,\widehat{g * e^{-ax^2}}(\xi) \\ &= \sqrt{\frac{a}{\pi}}\,\widehat{g * e^{-ax^2}}(\xi) \\ &= \frac{1}{\sqrt{4\pi kt}}\,\widehat{g * e^{-\frac{x^2}{4kt}}}(\xi)\end{aligned}$$

since $a = \frac{1}{4kt}$. Therefore

$$\widehat{\hat{g}(\xi)e^{-k\xi^2 t}} = \frac{1}{\sqrt{4\pi kt}}\, g * e^{-\frac{x^2}{4kt}} = \Phi(\cdot, t) * g = \int_{-\infty}^{\infty} \Phi(x-y,t)g(y)\,dy \tag{2}$$

where Φ is the fundamental solution of the heat equation in $\mathbb{R}$. Similarly

$$\widehat{e^{-k\xi^2(t-s)}\hat{f}(\xi,s)} = \frac{1}{\sqrt{4\pi k(t-s)}}\, e^{-\frac{x^2}{4k(t-s)}} * f(\cdot, s) = \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)\,dy. \tag{3}$$

Combining equations (1), (2) and (3) yields

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)g(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)f(y,s)\,dy\,ds$$

as required.

2. *The Fourier transform: The transport equation.*

(i) By definition

$$\begin{aligned}
\widehat{\tau_a v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau_a v(x) e^{-i\xi x}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x-a) e^{-i\xi x}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(y) e^{-i\xi(y+a)}\,dy \qquad (y = x-a) \\
&= e^{-i\xi a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(y) e^{-i\xi y}\,dy \\
&= e^{-i\xi a}\hat{v}(\xi)
\end{aligned}$$

as required.

(ii) Taking the Fourier transform of the transport equation $u_t + cu_x = 0$ gives

$$\widehat{u_t} + c\widehat{u_x} = 0 \quad \Longleftrightarrow \quad \hat{u}_t(\xi,t) + ci\xi\hat{u}(\xi,t) = 0$$

and taking the Fourier transform of the initial condition $u(x,0) = g(x)$ gives

$$\hat{u}(\xi,0) = \hat{g}(\xi).$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ:

$$\hat{u}_t = -ci\xi\hat{u}, \qquad \hat{u}(\xi,0) = \hat{g}(\xi).$$

Recall that the ODE $\dot{x} = \lambda x$ has solution $x(t) = x(0)e^{\lambda t}$. Applying this with $x = \hat{u}$, $\lambda = -ci\xi$ yields

$$\begin{aligned}
\hat{u}(\xi,t) &= \hat{u}(\xi,0)e^{-ci\xi t} \\
&= \hat{g}(\xi)e^{-ci\xi t} \\
&= \widehat{\tau_a g}(\xi)
\end{aligned}$$

where $a = ct$, by part (i). By taking the inverse Fourier transform we obtain

$$u(x,t) = \tau_a g(x) = g(x-a) = g(x-ct)$$

as desired.

3. *The Fourier transform: Schrödinger's equation.*

(i) Taking the Fourier transform of $iu_t = -u_{xx}$ with respect to the x variable gives

$$i\widehat{u_t} = -\widehat{u_{xx}} \quad \iff \quad i\hat{u}_t(\xi,t) = -(i\xi)^2\hat{u}(\xi,t) = \xi^2\hat{u}(\xi,t).$$

By multiplying by $-i$ we can rewrite this as $\hat{u}_t = -i\xi^2\hat{u}$. Taking the Fourier transform of the initial condition $u(x,0) = g(x)$ gives

$$\hat{u}(\xi,0) = \hat{g}(\xi).$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ:

$$\hat{u}_t = -i\xi^2\hat{u}, \qquad \hat{u}(\xi,0) = \hat{g}(\xi).$$

Recall that the ODE $\dot{x} = \lambda x$ has solution $x(t) = x(0)e^{\lambda t}$. Applying this with $x = \hat{u}$, $\lambda = -i\xi^2$ gives

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-i\xi^2 t} = \hat{g}(\xi)e^{-i\xi^2 t}. \tag{4}$$

To obtain u we need to compute the following inverse Fourier transform:

$$\widecheck{\hat{g}(\xi)e^{-i\xi^2 t}}.$$

The trick is to recognise that $\hat{g}(\xi)e^{-i\xi^2 t}$ is the product of Fourier transforms, which follows from the fact that the Fourier transform of a Gaussian is a Gaussian. Recall that

$$\widehat{e^{-ax^2}}(\xi) = \frac{1}{\sqrt{2a}}\, e^{-\frac{\xi^2}{4a}}.$$

Therefore

$$e^{-i\xi^2 t} = \sqrt{2a}\,\widehat{e^{-ax^2}}(\xi) \quad \text{for} \quad a = \frac{1}{4it}. \tag{5}$$

Since the product of Fourier transforms is the Fourier transform of a convolution, we obtain

$$\begin{aligned} \hat{g}(\xi)e^{-k\xi^2 t} &= \sqrt{2a}\,\hat{g}(\xi)\,\widehat{e^{-ax^2}}(\xi) && \text{(by equation (5))} \\ &= \sqrt{2a}\frac{1}{\sqrt{2\pi}}\,\widehat{g * e^{-ax^2}}(\xi) \\ &= \sqrt{\frac{a}{\pi}}\,\widehat{g * e^{-ax^2}}(\xi). \end{aligned}$$

Combining this with equation (4) and taking the inverse Fourier transform gives

$$\hat{u}(\xi,t) = \sqrt{\frac{a}{\pi}}\,\widehat{g * e^{-ax^2}}(\xi) \quad \iff \quad u(x,t) = \sqrt{\frac{a}{\pi}}\, g * e^{-ax^2}.$$

Since $a = \frac{1}{4it}$ and the convolution is commutative we arrive at

$$u(x,t) = \frac{1}{\sqrt{4\pi i t}}\, g * e^{-\frac{x^2}{4it}} = \frac{1}{\sqrt{4\pi i t}}\, e^{\frac{ix^2}{4t}} * g = \frac{1}{\sqrt{4\pi i t}} \int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{4t}} g(y)\, dy$$

as required.

(ii) In part (i) we showed that

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-i\xi^2 t}.$$

Since the Fourier transform preserves the L^2–norm,

$$\begin{aligned}\|u(\cdot,t)\|^2_{L^2(\mathbb{R})} &= \|\hat{u}(\cdot,t)\|^2_{L^2(\mathbb{R})} \\ &= \|\hat{g}(\xi)e^{-i\xi^2 t}\|^2_{L^2(\mathbb{R})} \\ &= \int_{-\infty}^{\infty} \left|\hat{g}(\xi)e^{-i\xi^2 t}\right|^2 d\xi \\ &= \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 \, d\xi \\ &= \|g\|^2_{L^2(\mathbb{R})}\end{aligned}$$

as required.

4. *The Fourier transform: The wave equation.* Use the Fourier transform to derive the solution

$$u(x,t) = \frac{1}{2}[g(x-ct) + g(x+ct)]$$

of the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) && \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= 0 && \text{for } x \in \mathbb{R}, \end{aligned}$$

where the constant $c > 0$ is the wave speed. This is known as *D'Alembert's solution.* Hint: Use Q2(i) and the fact that $\cos(c\xi t) = [\exp(ic\xi t) + \exp(-ic\xi t)]/2$.

Taking the Fourier transform of $u_{tt} = c^2 u_{xx}$ with respect to the x variable gives

$$\widehat{u_{tt}} = \widehat{c^2 u_{xx}} \quad \Longleftrightarrow \quad \hat{u}_{tt}(\xi,t) = c^2 (i\xi)^2 \hat{u}(\xi,t) = -c^2\xi^2\hat{u}(\xi,t).$$

Taking the Fourier transform of the initial condition $u(x,0) = g(x)$ gives

$$\hat{u}(\xi,0) = \hat{g}(\xi).$$

Taking the Fourier transform of the initial condition $u_t(x,0) = 0$ gives

$$\hat{u}_t(\xi,0) = 0.$$

We have reduced the PDE to a one-parameter family of uncoupled ODEs, indexed by ξ:

$$\hat{u}_{tt} = -c^2\xi^2\hat{u}, \qquad \hat{u}(\xi,0) = \hat{g}(\xi), \qquad \hat{u}_t(\xi,0) = 0.$$

Recall that the ODE $\ddot{x} = -\lambda^2 x$ has solution of the form $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$. Applying this with $x = \hat{u}$, $\lambda = c\xi$ gives

$$\hat{u}(\xi,t) = A\cos(c\xi t) + B\sin(c\xi t).$$

The initial conditions $\hat{u}(\xi,0) = \hat{g}(\xi)$, $\hat{u}_t(\xi,0) = 0$ imply that $A = \hat{g}(\xi)$ and $B = 0$. Therefore

$$\begin{aligned}\hat{u}(\xi,t) &= \hat{g}(\xi)\cos(c\xi t) \\ &= \hat{g}(\xi)\left[\frac{\exp(ic\xi t) + \exp(-ic\xi t)}{2}\right] \\ &= \frac{1}{2}\exp(ic\xi t)\hat{g}(\xi) + \frac{1}{2}\exp(-ic\xi t)\hat{g}(\xi) \\ &= \frac{1}{2}\widehat{\tau_{-a}g}(\xi) + \frac{1}{2}\widehat{\tau_a g}(\xi)\end{aligned}$$

where $a = ct$, by Q2(i). Taking the inverse Fourier transform gives

$$\begin{aligned} u(x,t) &= \frac{1}{2}\,\tau_{-a}\,g(x) + \frac{1}{2}\,\tau_a\,g(x) \\ &= \frac{1}{2}\,g(x+a) + \frac{1}{2}\,g(x-a) \\ &= \frac{1}{2}[g(x+ct) + g(x-ct)] \end{aligned}$$

as required.

5. *The Fourier transform of a derivative.* By definition

$$\begin{aligned} \widehat{u'}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u'(x) e^{-i\xi x}\,dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \frac{d}{dx} e^{-i\xi x}\,dx && \text{(integration by parts)} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)(-i\xi) e^{-i\xi x}\,dx \\ &= i\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-i\xi x}\,dx \\ &= i\xi \hat{u}(\xi) \end{aligned}$$

as required.

6. *The Fourier transform of a convolution.* By definition

$$\begin{aligned} \widehat{u * v}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u*v)(x) e^{-i\xi x}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(z) v(x-z)\,dz \right) e^{-i\xi x}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(z) v(x-z)\,dz \right) e^{-i\xi z} e^{-i\xi(x-z)}\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} v(x-z) e^{-i\xi(x-z)}\,dx \right) u(z) e^{-i\xi z}\,dz \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(\tilde{x}) e^{-i\xi\tilde{x}}\,d\tilde{x} \right) u(z) e^{-i\xi z}\,dz && (\tilde{x} = x - z) \\ &= \int_{-\infty}^{\infty} \hat{v}(\xi) u(z) e^{-i\xi z}\,dz \\ &= \sqrt{2\pi}\,\hat{v}(\xi) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(z) e^{-i\xi z}\,dz \\ &= \sqrt{2\pi}\,\hat{v}(\xi)\hat{u}(\xi) \end{aligned}$$

as desired.

7. *Proof of the Sobolev embedding using the Fourier transform.*

(i) Since the Fourier transform preserves the L^2–norm

$$\begin{aligned}
\|u\|_{H^1(\mathbb{R})}^2 &= \|u\|_{L^2(\mathbb{R})}^2 + \|u'\|_{L^2(\mathbb{R})}^2 \\
&= \|\hat{u}\|_{L^2(\mathbb{R})}^2 + \|\,\widehat{u'}\,\|_{L^2(\mathbb{R})}^2 \\
&= \|\hat{u}\|_{L^2(\mathbb{R})}^2 + \|i\xi\hat{u}\|_{L^2(\mathbb{R})}^2 \\
&= \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 \, d\xi + \int_{-\infty}^{\infty} |i\xi\hat{u}(\xi)|^2 \, d\xi \\
&= \int_{-\infty}^{\infty} (1+|\xi|^2)|\hat{u}(\xi)|^2 \, d\xi.
\end{aligned}$$

as required.

(ii) Following the hint

$$\begin{aligned}
\|\hat{u}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\hat{u}(\xi)| d\xi \\
&= \int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)^{1/2}} (1+|\xi|^2)^{1/2} \, |\hat{u}(\xi)| \, d\xi \\
&\le \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)} \, d\xi \right)^{1/2} \left(\int_{-\infty}^{\infty} (1+|\xi|^2) \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2} \qquad \text{(Cauchy-Schwarz)} \\
&= \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)} \, d\xi \right)^{1/2} \|u\|_{H^1(\mathbb{R})} \\
&= C\|u\|_{H^1(\mathbb{R})}
\end{aligned}$$

where

$$C = \left(\int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)} \, d\xi \right)^{1/2}.$$

If we can show that C is finite, then we've completed the proof. This is a simple calculus exercise; one way is as follows:

$$\begin{aligned}
C^2 &= 2\int_0^{\infty} \frac{1}{(1+\xi^2)} \, d\xi \\
&= \int_0^1 \frac{1}{(1+\xi^2)} \, d\xi + \int_1^{\infty} \frac{1}{(1+\xi^2)} \, d\xi \\
&\le \int_0^1 \frac{1}{(1+0)} \, d\xi + \int_1^{\infty} \frac{1}{\xi^2} \, d\xi \\
&= 1 + 1 \\
&= 2 < \infty
\end{aligned}$$

as required.

(iii) By the Fourier Inversion Theorem

$$
\begin{aligned}
|u(x)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{i\xi x}\, d\xi \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{u}(\xi)|\, |e^{i\xi x}|\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{u}(\xi)|\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \|\hat{u}\|_{L^1(\mathbb{R})} \\
&\leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^1(\mathbb{R})}
\end{aligned}
$$

by part (ii). Since this holds for all $x \in \mathbb{R}$,

$$
\|u\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{2\pi}} \|u\|_{H^1(\mathbb{R})}
$$

as required.

8. *Fundamental Solution of the Heat Equation.* We compute

$$
\begin{aligned}
\Phi_t(\boldsymbol{x}, t) &= \frac{1}{(4\pi k)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}} \left(-\frac{n}{2} t^{-\frac{n}{2}-1} + \frac{|\boldsymbol{x}|^2}{4kt^2} \right) \\
\frac{\partial \Phi}{\partial x_i}(\boldsymbol{x}, t) &= \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}} \left(-\frac{2x_i}{4kt} \right), \\
\frac{\partial^2 \Phi}{\partial x_i^2}(\boldsymbol{x}, t) &= \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}} \left(-\frac{2}{4kt} + \left(\frac{2x_i}{4kt} \right)^2 \right) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}} \left(-\frac{1}{2kt} + \frac{x_i^2}{4k^2t^2} \right), \\
\Delta\Phi(\boldsymbol{x}, t) &= \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2}(\boldsymbol{x}, t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4kt}} \left(-\frac{n}{2kt} + \frac{|\boldsymbol{x}|^2}{4k^2t^2} \right).
\end{aligned}
$$

Therefore Φ satisfies $\Phi_t(\boldsymbol{x}, t) = k\Delta\Phi(\boldsymbol{x}, t)$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $t > 0$, as required.

9. *Finite speed of propagation for a degenerate diffusion equation.*

(i) Observe that

$$
\frac{1}{2} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} - \frac{1}{6k} \frac{x^2}{t} = 0 \quad \Longleftrightarrow \quad |x| = 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}
$$

and so

$$
\Phi(x, t) = \begin{cases} \frac{1}{2} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} - \frac{1}{6k} \frac{x^2}{t} & \text{if } |x| < 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}, \\ 0 & \text{if } |x| \geq 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}. \end{cases}
$$

Clearly Φ satisfies the degenerate diffusion equation for $|x| > 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$. For $|x| < 3^{\frac{2}{3}} k^{\frac{1}{3}} \pi^{-\frac{1}{6}} t^{\frac{1}{3}}$,

$$
\begin{aligned}
\Phi_t(x, t) &= \frac{1}{6} \left(\frac{3}{k\pi t} \right)^{-\frac{2}{3}} \left(-\frac{3}{k\pi t^2} \right) + \frac{x^2}{6kt^2} = -\frac{1}{6t} \left(\frac{3}{k\pi t} \right)^{\frac{1}{3}} + \frac{x^2}{6kt^2}, \\
\Phi_x(x, t) &= -\frac{x}{3kt},
\end{aligned}
$$

$$(a(\Phi)\Phi_x)(x,t) = -\frac{x}{6t}\left(\frac{3}{k\pi t}\right)^{\frac{1}{3}} + \frac{x^3}{18kt^2},$$
$$(a(\Phi)\Phi_x)_x(x,t) = -\frac{1}{6t}\left(\frac{3}{k\pi t}\right)^{\frac{1}{3}} + \frac{x^2}{6kt^2}.$$

Therefore Φ satisfies the degenerate diffusion equation for all $t > 0$ and all $x \in \mathbb{R}$ with $|x| \neq 3^{\frac{2}{3}}k^{\frac{1}{3}}\pi^{-\frac{1}{6}}t^{\frac{1}{3}}$.

(ii) For fixed $t > 0$, $\Phi(x,t)$ is the maximum of a concave quadratic function and the zero function. The support of the map $x \mapsto \Phi(x,t)$ is the compact interval

$$[-3^{\frac{2}{3}}k^{\frac{1}{3}}\pi^{-\frac{1}{6}}t^{\frac{1}{3}}, 3^{\frac{2}{3}}k^{\frac{1}{3}}\pi^{-\frac{1}{6}}t^{\frac{1}{3}}].$$

10. *The mathematical equation that caused the banks to crash.* Let $t(x,\tau)$ satisfy

$$\tau = T - t(x,\tau).$$

Differentiating this expression with respect to τ and x gives

$$t_\tau = -1, \qquad t_x = 0.$$

Let $S(x,\tau)$ satisfy

$$x = \ln\left(\frac{S(x,\tau)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau. \tag{6}$$

Differentiating this expressions with respect to τ gives

$$0 = \frac{K}{S}\frac{S_\tau}{K} + r - \frac{1}{2}\sigma^2 \quad \Longleftrightarrow \quad S_\tau = \left(\frac{1}{2}\sigma^2 - r\right)S.$$

Differentiating equation (6) with respect to x gives

$$1 = \frac{K}{S}\frac{S_x}{K} \quad \Longleftrightarrow \quad S_x = S.$$

Therefore

$$\begin{aligned}
u_\tau &= Ce^{r\tau}\left[rV + V_S S_\tau + V_t t_\tau\right] = Ce^{r\tau}\left[rV + \left(\frac{1}{2}\sigma^2 - r\right)SV_s - V_t\right],\\
u_x &= Ce^{r\tau}\left[V_S S_x + V_t t_x\right] = Ce^{r\tau}SV_S,\\
u_{xx} &= Ce^{r\tau}\left[S_x V_S + SV_{SS}S_x\right] = Ce^{r\tau}\left[SV_S + S^2V_{SS}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
u_\tau - \frac{1}{2}\sigma^2 u_{xx} &= Ce^{r\tau}\left[rV + \left(\frac{1}{2}\sigma^2 - r\right)SV_s - V_t - \frac{1}{2}\sigma^2(SV_S + S^2V_{SS})\right]\\
&= -Ce^{r\tau}\left[V_t + \frac{1}{2}\sigma^2S^2V_{SS} + rSV_S - rV\right]\\
&= 0
\end{aligned}$$

since V satisfies the Black-Scholes PDE. This completes the proof.

11. *The energy method: Uniqueness for the heat equation in a time dependent domain.* Let u and v be solutions and let $w = u - v$. Then w satisfies

$$\begin{aligned} w_t - kw_{xx} &= 0 \quad \text{for } (x,t) \in U, \\ w(a(t),t) &= 0 \quad \text{for } t \in [0,T], \\ w(b(t),t) &= 0 \quad \text{for } t \in [0,T], \\ w(x,0) &= 0 \quad \text{for } x \in (a(0),b(0)). \end{aligned}$$

Multiply the equation $w_t = kw_{xx}$ by w and integrate over $(a(t),b(t))$ to obtain

$$\int_{a(t)}^{b(t)} ww_t\,dx = k\int_{a(t)}^{b(t)} ww_{xx}\,dx. \tag{7}$$

Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} f_t(x,t)\,dx + \dot{b}(t)f(b(t),t) - \dot{a}(t)f(a(t),t).$$

We can use this to rewrite the left-hand side of equation (7) as follows:

$$\begin{aligned} \int_{a(t)}^{b(t)} ww_t\,dx &= \frac{d}{dt}\frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx - \frac{1}{2}\dot{b}(t)w^2(b(t),t) + \frac{1}{2}\dot{a}(t)w^2(a(t),t) \\ &= \frac{d}{dt}\frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx \end{aligned} \tag{8}$$

since $w(a(t),t) = w(b(t),t) = 0$. We rewrite the right-hand side of equation (7) using integration by parts:

$$k\int_{a(t)}^{b(t)} ww_{xx}\,dx = kww_x\Big|_{a(t)}^{b(t)} - k\int_{a(t)}^{b(t)} w_x^2\,dx = -k\int_{a(t)}^{b(t)} w_x^2\,dx, \tag{9}$$

again using the fact that $w(a(t),t) = w(b(t),t) = 0$. Substituting (8) and (9) into (7) gives

$$\frac{d}{dt}\frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx = -k\int_{a(t)}^{b(t)} w_x^2\,dx \le 0.$$

Let

$$E(t) = \frac{1}{2}\int_{a(t)}^{b(t)} w^2(x,t)\,dx.$$

We have shown that $\dot{E}(t) \le 0$. Hence

$$0 \le E(t) \le E(0) = 0$$

since $w = 0$ for $t = 0$. Consequently $E(t) = 0$ for all t. Therefore $w = 0$ in U and so $u = v$, as required.

12. *The energy method: Uniqueness for a 4th-order heat equation.* Let u and v be solutions and let $w = u - v$. Then w satisfies

$$w_t + kw_{xxxx} = 0 \quad \text{for } (x,t) \in (a,b) \times (0,T], \tag{10}$$
$$w(a,t) = w(b,t) = 0 \quad \text{for } t \in [0,T], \tag{11}$$
$$w_x(a,t) = w_x(b,t) = 0 \quad \text{for } t \in [0,T], \tag{12}$$
$$w(x,0) = 0 \quad \text{for } x \in (a,b). \tag{13}$$

Multiplying the equation $w_t = -kw_{xxxx}$ by w and integrating over (a, b) gives

$$\begin{aligned}
\int_a^b \underbrace{ww_t}_{\frac{1}{2}\frac{\partial}{\partial t}w^2} \, dx &= -k \int_a^b ww_{xxxx} \, dx \\
&= -kww_{xxx}\Big|_a^b + k \int_a^b w_x w_{xxx} \, dx \\
&= k \int_a^b w_x w_{xxx} \, dx && \text{(by (11))} \\
&= kw_x w_{xx}\Big|_a^b - k \int_a^b w_{xx} w_{xx} \, dx \\
&= -k \int_a^b w_{xx}^2 \, dx && \text{(by (12))}.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\frac{1}{2}\int_a^b w^2 \, dx = -k \int_a^b w_{xx}^2 \, dx \le 0$$

and so

$$0 \le \int_a^b w^2(x,t) \, dx \le \int_a^b w^2(x,0) \, dx = 0$$

by (13). We conclude that $w = 0$ and hence $u = v$, as required.

We consider $u_t + ku_{xxxx} = 0$ to be the 4th-order version of the heat equation $u_t - ku_{xx} = 0$ since it has the same energy-decay property:

$$\frac{d}{dt}\|u\|^2_{L^2([a,b])} \le 0$$

(provided that u and u_x vanish at $x = a$ and $x = b$). The equation $u_t - ku_{xxxx} = 0$ looks more similar to the heat equation $u_t - ku_{xx} = 0$ (because of the minus sign), but its L^2–energy grows with time:

$$\frac{d}{dt}\|u\|^2_{L^2([a,b])} \ge 0.$$

13. *Asymptotic behaviour of the heat equation with time independent data.* Let $w(\boldsymbol{x}, t) = u(\boldsymbol{x}, t) - v(\boldsymbol{x})$. We need to prove that $\lim_{t\to\infty} \|w\|_{L^2(\Omega)} = 0$. By subtracting the PDEs for u and v we find that w satisfies

$$\begin{aligned}
w_t(\boldsymbol{x}, t) - k\Delta w(\boldsymbol{x}, t) &= 0 \quad \text{for } (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\
w(\boldsymbol{x}, t) &= 0 \quad \text{for } (\boldsymbol{x}, t) \in \partial\Omega \times [0, \infty), \\
w(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}) - v(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega.
\end{aligned}$$

Multiplying the equation $w_t = k\Delta w$ by w and integrating by parts over Ω gives

$$\begin{aligned}
\int_\Omega ww_t \, d\boldsymbol{x} = k \int_\Omega w\Delta w \, d\boldsymbol{x} \quad &\iff \quad \frac{d}{dt}\frac{1}{2}\int_\Omega w^2 \, d\boldsymbol{x} = k \int_{\partial\Omega} w \, \nabla w \cdot \boldsymbol{n} \, dS - k \int_\Omega \nabla w \cdot \nabla w \, d\boldsymbol{x} \\
&\iff \quad \frac{d}{dt}\frac{1}{2}\int_\Omega w^2 \, d\boldsymbol{x} = -k \int_\Omega |\nabla w|^2 \, d\boldsymbol{x} \qquad (14)
\end{aligned}$$

since $w = 0$ on $\partial\Omega$. By the Poincaré inequality, there exists a constant $C_p > 0$ such that

$$\int_\Omega |w|^2 \, d\boldsymbol{x} \le C_p \int_\Omega |\nabla w|^2 \, d\boldsymbol{x}.$$

Multiplying this by $-k/C_p$ gives

$$-\frac{k}{C_p}\int_\Omega |w|^2\,d\boldsymbol{x} \geq -k\int_\Omega |\nabla w|^2\,d\boldsymbol{x}. \tag{15}$$

Combining equations (14), (15) yields

$$\frac{d}{dt}\frac{1}{2}\int_\Omega w^2\,d\boldsymbol{x} \leq -\frac{k}{C_p}\int_\Omega |w|^2\,d\boldsymbol{x}.$$

Define

$$E(t) = \int_\Omega w^2(\boldsymbol{x},t)\,d\boldsymbol{x} = \|w\|^2_{L^2(\Omega)}$$

and $\lambda = \frac{2k}{C_p}$. We have shown that

$$\dot{E} \leq -\lambda E.$$

By the Gr̈onwall inequality,

$$E(t) \leq e^{-\lambda t}E(0).$$

Since $\lambda > 0$, we conclude that $E(t) \to 0$ as $t \to \infty$. Therefore $w \to 0$ in $L^2(\Omega)$ as $t \to \infty$, as required.

14. *Asymptotic behaviour of the heat equation with time independent data in the L^∞–norm.*

(i) By linearity, w satisfies

$$\begin{aligned} w_t(x,t) - kw_{xx}(x,t) &= 0 \quad \text{for } (x,t) \in (a,b)\times(0,\infty), \\ w(x,0) &= u_0(x) - v(x) \quad \text{for } x \in (a,b), \\ w(a,t) = w(b,t) &= 0 \quad \text{for } t \in [0,\infty). \end{aligned}$$

Multiplying the PDE by w and integrating over $[a,b]$ gives

$$\int_a^b ww_t\,dx = k\int_a^b ww_{xx}\,dx \quad\Longleftrightarrow\quad \frac{d}{dt}\frac{1}{2}\int_a^b w^2\,dx = \underbrace{kww_x\Big|_a^b}_{=0} - k\int_a^b w_x^2\,dx$$

where we have used the Chain Rule and integration by parts. Note that the boundary terms vanish when we perform integration by parts since $w(a,t) = w(b,t) = 0$. Therefore

$$\frac{d}{dt}\int_a^b w^2(x,t)\,dx = -2k\int_a^b w_x^2(x,t)\,dx$$

as required.

(ii) We can write the result of part (i) in terms of L^2–norms as

$$\frac{d}{dt}\|w\|^2_{L^2([a,b])} = -2k\|w_x\|^2_{L^2([a,b])}. \tag{16}$$

By the Poincaré inequality, there exists a constant $C > 0$ such that

$$\|w\|^2_{L^2([a,b])} \leq C\|w_x\|^2_{L^2([a,b])}. \tag{17}$$

Combining equations (16), (17) gives

$$\frac{d}{dt}\|w\|^2_{L^2([a,b])} = -2k\|w_x\|^2_{L^2([a,b])} \leq -\frac{2k}{C}\|w\|^2_{L^2([a,b])}. \tag{18}$$

Define

$$E(t) = \|w\|^2_{L^2([a,b])}.$$

We can rewrite equation (18) as

$$\dot{E} \le -\lambda E$$

with $\lambda = 2k/C > 0$. By the Grönwall inequality,

$$E(t) \le E(0)e^{-\lambda t} \to 0 \quad \text{as } t \to \infty.$$

Therefore, by definition of E, $w \to 0$ in $L^2([a,b])$ as $t \to \infty$, as required.

(iii) By differentiating the PDE for w with respect to t we obtain

$$\begin{aligned} w_{tt}(x,t) - kw_{txx}(x,t) &= 0 \quad \text{for } (x,t) \in (a,b) \times (0,\infty), \\ w_t(a,t) = w_t(b,t) &= 0 \quad \text{for } t \in [0,\infty). \end{aligned}$$

In particular, w_t satisfies the heat equation with Dirichlet boundary conditions, just like w. Therefore the argument we applied in parts (i) and (ii) to w can also be applied to w_t, which yields $w_t \to 0$ in $L^2([a,b])$ as $t \to \infty$.

(iv) We have

$$\begin{aligned} \|w_x\|^2_{L^2([a,b])} &= \int_a^b w_x^2(x,t)\,dx \\ &= -\frac{1}{2k}\frac{d}{dt}\int_a^b w^2(x,t)\,dx && \text{(by part (i))} \\ &= -\frac{1}{k}\int_a^b w(x,t)w_t(x,t)\,dx \\ &\le \frac{1}{k}\left(\int_a^b w^2(x,t)\,dx\right)^{1/2}\left(\int_a^b w_t^2(x,t)\,dx\right)^{1/2} && \text{(Cauchy-Schwarz)} \\ &= \frac{1}{k}\|w\|_{L^2([a,b])}\|w_t\|_{L^2([a,b])} \to 0 \quad \text{as } t \to \infty \end{aligned}$$

by parts (ii) and (iii). Therefore $w_x \to 0$ in $L^2([a,b])$ as $t \to \infty$, as required. Note that we don't really need $w_t \to 0$ in $L^2([a,b])$ as $t \to \infty$, we just need $\|w_t\|_{L^2([a,b])}$ to be uniformly bounded in t.

(v) This final result follows from the Sobolev inequality: There exists a constant $C > 0$ such that

$$\|w\|_{L^\infty([a,b])} \le C\|w\|_{H^1([a,b])} = C\left(\|w\|_{L^2([a,b])} + \|w_x\|_{L^2([a,b])}\right)^{1/2} \to 0 \quad \text{as } t \to \infty$$

by parts (ii) and (iv).

15. *Applications of the maximum principle: Uniqueness and bounds on solutions.*

(i) Let $\Gamma_T = [a,b] \times \{0\} \cup \{a,b\} \times [0,T]$ be the parabolic boundary of Ω_T. Let $u, v \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfy

$$\begin{aligned} u_t - u_{xx} &= 1 \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{in } \Gamma_T. \end{aligned}$$

Then $w = u - v \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 \quad \text{in } \Omega_T, \\ w &= 0 \quad \text{in } \Gamma_T. \end{aligned}$$

By the weak maximum principle

$$\max_{\overline{\Omega}_T} w = \max_{\Gamma_T} w = 0, \qquad \min_{\overline{\Omega}_T} w = \min_{\Gamma_T} w = 0.$$

Therefore $w = 0$ and $u = v$, as required.

(ii) Since $u_t - u_{xx} = 1 > 0$, the weak maximum principle gives

$$\min_{\overline{\Omega}_T} u = \min_{\Gamma_T} u = 0.$$

This is the desired lower bound on u. We still need to prove the upper bound. Let $v(x,t) = t$. Then $v_t - v_{xx} = 1$ and $w = u - v$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 \quad \text{in } \Omega_T, \\ w &= -t \quad \text{in } \Gamma_T. \end{aligned}$$

By the weak maximum principle

$$\max_{\overline{\Omega}_T} w = \max_{\Gamma_T} w = \max_{\Gamma_T}(-t) = 0.$$

Therefore $w \le 0$ in Ω_T and hence $u \le v = t$ in Ω_T, which is the desired upper bound.

16. *Application of the maximum principle: Comparison Principle.* Define $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} \frac{\partial v}{\partial t}(\boldsymbol{x},t) - k\Delta v(\boldsymbol{x},t) &= f_1(\boldsymbol{x}) - f_2(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x},t) \in \Omega \times (0,T], \\ v(\boldsymbol{x},t) &= g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}) \quad \text{for } (\boldsymbol{x},t) \in \partial\Omega \times [0,T], \\ v(\boldsymbol{x},0) &= u_1^0(\boldsymbol{x}) - u_2^0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega. \end{aligned}$$

Since $f_1 \le f_2$, then

$$v_t - k\Delta v = f_1 - f_2 \le 0 \quad \text{in } \Omega_T.$$

Therefore the weak maximum principle implies that

$$\max_{\overline{\Omega}_T} v = \max_{\Gamma_T} v.$$

For $(\boldsymbol{x},t) \in \Gamma_T$,

$$v(\boldsymbol{x},t) = \begin{cases} g_1(\boldsymbol{x}) - g_2(\boldsymbol{x}) & \text{if } (\boldsymbol{x},t) \in \partial\Omega \times [0,T], \\ u_1^0(\boldsymbol{x}) - u_2^0(\boldsymbol{x}) & \text{if } t = 0,\ \boldsymbol{x} \in \Omega. \end{cases}$$

But

$$g_1 - g_2 \le 0, \qquad u_1^0 - u_2^0 \le 0.$$

Therefore $v \le 0$ on Γ_T and hence

$$\max_{\overline{\Omega}_T} v = \max_{\Gamma_T} v \le 0.$$

Hence $v \le 0$ in Ω_T and so $u_1 \le u_2$ in Ω_T, as required.

17. *Eigenfunctions of the Laplacian and an application to the heat equation.* Formally (not worrying about interchanging limits and infinite sums),

$$\begin{aligned}
0 &= v_t - k\Delta v \\
&= \sum_{n=1}^{\infty} \dot{c}_n(t) u_n(\boldsymbol{x}) - k \sum_{n=1}^{\infty} c_n(t) \Delta u_n(\boldsymbol{x}) \\
&= \sum_{n=1}^{\infty} \dot{c}_n(t) u_n(\boldsymbol{x}) + k \sum_{n=1}^{\infty} c_n(t) \lambda_n u_n(\boldsymbol{x}) \\
&= \sum_{n=1}^{\infty} \left(\dot{c}_n(t) + k\lambda_n c_n(t)\right) u_n(\boldsymbol{x}).
\end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ forms an orthogonal basis, it follows that

$$\dot{c}_n(t) + k\lambda_n c_n(t) = 0$$

for all n. We also have

$$v(\boldsymbol{x}, 0) = g(\boldsymbol{x}) \quad \iff \quad \sum_{n=1}^{\infty} c_n(0) u_n(\boldsymbol{x}) = \sum_{n=1}^{\infty} g_n u_n(\boldsymbol{x}) \quad \iff \quad \sum_{n=1}^{\infty} (c_n(0) - g_n) u_n(\boldsymbol{x}) = 0.$$

Again, since $\{u_n\}_{n\in\mathbb{N}}$ forms an orthogonal basis, it follows that

$$c_n(0) = g_n$$

for all n. We have reduced the PDE for v to a one-parameter family of uncoupled ODEs, indexed by n:

$$\dot{c}_n(t) = -k\lambda_n c_n(t), \qquad c_n(0) = g_n.$$

These ODEs have solutions

$$c_n(t) = g_n e^{-k\lambda_n t}.$$

Therefore

$$v(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} g_n e^{-k\lambda_n t} u_n(\boldsymbol{x})$$

as required.