Additional Exercise - Wave Equation Solution

Exercise. Consider the wave equation

$$(1)\qquad \begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & (x,t) \in (0,2\pi) \times (0,\infty), \\ u(0,t) = u(2\pi,t), & t \in (0,\infty), \\ u(x,0) = f(x), & t \in (0,2\pi), \\ u_t(x,0) = 0, & x \in (0,2\pi), \end{cases}$$

where f is a smooth function on $[0,2\pi]$ with $f(0) = f(2\pi) = 0$. We will find a solution to the (1) by utilising the Fourier series (like the heat equation). We assume that we can write

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(t) e^{inx}$$

with

$$a_n(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx} dx$$

and that the convergence is such that all interchanging of differentiation and integration is allowed (this will happen, for instance, if we seek a solution that is C^4 on the periodic domain).

(i) Show that $a_n(t)$ satisfies the equation

$$a_n''(t) + n^2 c^2 a_n(t) = 0.$$

(ii) using he boundary conditions show that

$$a_n(t) = A_n \cos(nct)$$

and find an explicit expression for $\{A_n\}_{n\in\mathbb{Z}}$ which depends on f.

(iii) Write a solution to (1)

Solution. (i) Differentiating the sum and plugging it into our wave equation yields

$$\sum_{n\in\mathbb{Z}} a_n''(t) e^{inx} = c^2 \sum_{n\in\mathbb{Z}} a_n(t) (in)^2 e^{inx}.$$

The uniqueness of the Fourier coefficients implies that $a_n(t)$ must solve the equation

$$a_n''(t) = -n^2 c^2 a_n(t),$$

which is the desired result.

(ii) The solution to the above sequence of ODEs is given by

$$a_n(t) = \begin{cases} A_0 + B_0 t, & \text{n=0,} \\ A_n \cos(nct) + B_n \sin(nct), & n \neq 0 \end{cases}.$$

At this point we will need to use our boundary condition for $u(x,0)$ and $u_t(x,0)$. Since

$$u_t(x,t) = \sum_{n\in\mathbb{Z}} a_n'(t)e^{inx}$$

we conclude that $u_t(x,0) = 0$ implies, together with the uniqueness of the Fourier coefficients, that

$$a_n'(0) = 0, \qquad \forall n \in \mathbb{Z}.$$

As

$$a_n'(t) = \begin{cases} B_0, & \text{n=0,} \\ -ncA_n \sin(nct) + ncB_n \cos(nct), & n \neq 0 \end{cases}.$$

we find that $B_n = 0$ for all $n \in \mathbb{Z}$. Thus, we have that

$$a_n(t) = \begin{cases} A_0, & n = 0, \\ A_n \cos(nct), & n \neq 0. \end{cases} = A_n \cos(nct).$$

Using the fact that $u(x,0) = f(x)$ we find that

$$\sum_{n\in\mathbb{Z}} a_n(0)e^{inx} = u(x,0) = f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{inx}$$

and, like before, the uniqueness of the Fourier coefficients imply that

$$A_n = a_n(0) = \widehat{f}_n, \qquad \forall n \in \mathbb{Z}.$$

(iii) Combining the previous results we conclude that our solution is given by

$$u(x,t) = \sum_{n\in\mathbb{Z}} \widehat{f}_n \cos(nct)\, e^{inx}$$

□