Problem Class 1

Problem 1: For

$$\Lambda_{1/2} = \exp\left(S^{\mu\nu}\theta_{\mu\nu}\right) \tag{0.1}$$

and show that $\Lambda_{1/2}^\dagger \neq \Lambda_{1/2}^{-1}$.

solution: This is equivalent to $(S^{\mu\nu}\theta_{\mu\nu})^\dagger = -S^{\mu\nu}\theta_{\mu\nu}$. Note that using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ implies that whenever $\mu \neq \nu$:

$$S^{\mu\nu} = \tfrac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = \tfrac{1}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = \tfrac{1}{4}\left(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu\right) = \tfrac{1}{2}\gamma^\mu\gamma^\nu\,. \tag{0.2}$$

Furthermore, the concrete way in which we have been writing the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ -\mathbb{1}_{2\times 2} & 0 \end{pmatrix}\,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \;\; i = 1,2,3 \tag{0.3}$$

shows that $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^i)^\dagger = \gamma^i$. Now let's go: Letting $i, j = 1, 2, 3$, we can work this out as

$$\begin{aligned} (S^{0i})^\dagger &= \tfrac{1}{2}(\gamma^0\gamma^i)^\dagger = \tfrac{1}{2}(\gamma^i)^\dagger(\gamma^0)^\dagger = -\tfrac{1}{2}(\gamma^i)(\gamma^0) = S^{0i} \\ (S^{ij})^\dagger &= \tfrac{1}{2}(\gamma^i\gamma^j)^\dagger = \tfrac{1}{2}(\gamma^j)^\dagger(\gamma^i)^\dagger = \tfrac{1}{2}(\gamma^j)(\gamma^i) = -S^{ij}\,. \end{aligned} \tag{0.4}$$

Hence for a general choice of the real numbers $\theta_{\mu\nu}$ we have

$$(S^{\mu\nu}\theta_{\mu\nu})^\dagger \neq -S^{\mu\nu}\theta_{\mu\nu} \tag{0.5}$$

so that $\Lambda_{1/2}^\dagger \neq \Lambda_{1/2}^{-1}$.

Problem 2: Show that

$$\Lambda_{1/2}^\dagger\gamma^0 = \gamma^0\Lambda_{1/2}^{-1} \tag{0.6}$$

solution: We can work this out as follows

$$\Lambda_{1/2}^\dagger\gamma^0 = \left(\exp\left(S^{\mu\nu}\theta_{\mu\nu}\right)\right)^\dagger\gamma^0 = \sum_{k=0}^{\infty}\frac{1}{k!}((S^{\mu\nu})^\dagger\theta_{\mu\nu})^k\gamma^0 \tag{0.7}$$

so we need to know how to commute γ^0 with $(S^{\mu\nu})^\dagger$. Note that

$$(S^{\mu\nu})^\dagger\gamma^0 = -\gamma^0 S^{\mu\nu}\,, \tag{0.8}$$

when $\mu = 0$ and $\nu = 1, 2, 3$, we get no minus sign from the † and one minus sign from $\gamma^i \gamma^0 = -\gamma^0 \gamma^i$, when $\mu, \nu \neq 0$ we get a minus sign from the † and two minus signs from pulling γ^0 to the left. We can now work out

$$\begin{aligned}((S^{\mu\nu})^\dagger \theta_{\mu\nu})^k \gamma^0 = ((S^{\mu\nu})^\dagger \theta_{\mu\nu})^{k-1} \gamma^0 (-S^{\mu\nu}\theta_{\mu\nu}) = ((S^{\mu\nu})^\dagger \theta_{\mu\nu})^{k-2} \gamma^0 (-S^{\mu\nu}\theta_{\mu\nu})^2 = \\ ... = \gamma^0 (-S^{\mu\nu}\theta_{\mu\nu})^k\end{aligned} \tag{0.9}$$

Hence

$$\Lambda_{1/2}^\dagger \gamma^0 = \sum_{k=0}^{\infty} \frac{1}{k!} ((S^{\mu\nu})^\dagger \theta_{\mu\nu})^k \gamma^0 = \sum_{k=0}^{\infty} \frac{1}{k!} \gamma^0 (-S^{\mu\nu}\theta_{\mu\nu})^k = \gamma^0 \Lambda_{1/2}^{-1} \tag{0.10}$$

and we are done.