Problem Class 2

Problem 1: The action

$$S = \int dt d^3x \quad - |\boldsymbol{\nabla}\psi|^2 + \tfrac{1}{2} i \left(\bar{\psi}\partial_t \psi - \psi \partial_t \bar{\psi} \right) \tag{0.1}$$

has a $U(1)$ symmetry $\psi \to e^{i\theta}\psi$, $\bar{\psi} \to e^{-i\theta}\bar{\psi}$, $\theta \in \mathbb{R}$. Find the associated conserved current.

solution:

First note that we should treat ψ and $\bar{\psi}$ as independent fields. There is only a single Lie algebra element $\gamma = i\theta$ to consider and

$$\delta_\gamma \psi = i\theta\psi \qquad \delta_\gamma \bar{\psi} = -i\theta\bar{\psi} \tag{0.2}$$

and

$$\begin{aligned} j^0 &= i\theta\psi \frac{\partial \mathcal{L}}{\partial \partial_t \psi} - i\theta\bar{\psi}\frac{\partial \mathcal{L}}{\partial \partial_t \bar{\psi}} = -|\psi|^2 \\ j^j &= i\theta\psi \frac{\partial \mathcal{L}}{\partial \partial_j \psi} - i\theta\bar{\psi}\frac{\partial \mathcal{L}}{\partial \partial_j \bar{\psi}} \\ &= i\theta\psi(-\partial_j\bar{\psi}) - i\theta\bar{\psi}(-\partial_j\psi) \\ &= -i\theta\left(\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi\right) \end{aligned} \tag{0.3}$$

and the conservation equation is

$$\partial_t |\psi|^2 + \partial_j \left(i\psi\partial_j\bar{\psi} - i\bar{\psi}\partial_j\psi \right) = 0 \tag{0.4}$$

This equation guarantees that the probability density $|\psi|^2$ is conserved in quantum mechanics:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} d^3x |\psi|^2 = 0\,, \tag{0.5}$$

and what we have just seen is that this is enforced by a $U(1)$ symmetry!

Problem 2: Consider the following action of a complex scalar field:

$$\mathcal{L} = \partial_\mu\bar{\phi}\partial^\mu\phi + m^2|\phi|^2\,. \tag{0.6}$$

a) Show that this Lagrangian is Lorentz invariant.

b) Show the equation of motion

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0\,.$$

is Lorentz invariant.

c) Find the conserved current j^μ associated to a $U(1)$ symmetry of $\mathcal{L}$ acting as $\phi \to e^{i\theta}\phi$, $\theta \in \mathbb{R}$.

d) Show that j^μ is real. Is j^0 always positive? [hint: try plane wave solutions $\phi = \exp\{ik_\mu x^\mu\}$].

solution:

a) We work out

$$\begin{aligned}\mathcal{L} \to\ & \partial_\mu \bar{\phi}'(\boldsymbol{x})\partial^\mu \phi'(\boldsymbol{x}) + m^2|\phi'|^2(\boldsymbol{x}) \\ &= \eta_{\nu\mu}\Lambda^\nu{}_\rho \Lambda^\mu{}_\rho (\partial_y)^\rho \bar{\phi}(\Lambda^{-1}\boldsymbol{x})(\partial_y)^\sigma \phi(\Lambda^{-1}\boldsymbol{x}) + m^2|\phi'|^2(\Lambda^{-1}\boldsymbol{x}) \\ &= (\partial_y)_\mu \bar{\phi}(\Lambda^{-1}\boldsymbol{x})(\partial_y)^\mu \phi(\Lambda^{-1}\boldsymbol{x}) + m^2|\phi|^2(\Lambda^{-1}\boldsymbol{x})\end{aligned} \tag{0.7}$$

where $\partial_\mu = \partial/\partial x^\mu$ and $(\partial_y)_\mu = \partial/\partial y^\mu$.

Hence all that is happening here is that we move the entire action from $\boldsymbol{x}$ to $\Lambda^{-1}\boldsymbol{x}$, which is what we defined as the behavior of a Lorentz invariant Lagrangian:

$$\mathcal{L}(\phi(\boldsymbol{x}), \partial/\partial x^\mu \phi(\boldsymbol{x})) \mapsto \mathcal{L}(\phi(\boldsymbol{y}), \partial/\partial y^\mu \phi(\boldsymbol{y})) \tag{0.8}$$

with $\boldsymbol{y} = \Lambda^{-1}\boldsymbol{x}$. What made that work was the $\partial_\mu\phi\partial^\mu\phi$ for which the 'exterior' Λ cancelled out.

b) Let's look at the equations of motion from problem 10:

$$0 = (\partial_\mu\partial^\mu - m^2)\phi(\boldsymbol{x}) = \eta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - m^2\right)\phi(\boldsymbol{x})\,. \tag{0.9}$$

Let us assume that we have found a solution $\phi_0(\boldsymbol{x})$. Then an argument as done in the lecture shows that

$$\begin{aligned}0 =& \eta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - m^2\right)\phi_0(\boldsymbol{x}) \\ =& \eta^{\mu\nu}\left(\Lambda^{-1}\right)^\rho{}_\mu \left(\Lambda^{-1}\right)^\sigma{}_\nu \left(\frac{\partial}{\partial y^\rho}\frac{\partial}{\partial y^\sigma} - m^2\right)\phi_0(\boldsymbol{y})\Big|_{\boldsymbol{y}=\Lambda^{-1}\boldsymbol{x}} \\ =& \left(\eta_{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu} - m^2\right)\phi_0(\boldsymbol{y})\Big|_{\boldsymbol{y}=\Lambda^{-1}\boldsymbol{x}}\end{aligned} \tag{0.10}$$

Here we used that $\Lambda^T\eta\Lambda = \eta$ holds for Lorentz transformations (and hence for inverses of Lorentz transformations). As this equation must be true for all $\boldsymbol{x}$, it is also true for all $\boldsymbol{y}$, in other words $\phi_0(\boldsymbol{y})$ is also a solution to the equations of motion.

c) We have

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\gamma \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \delta_\gamma \bar{\phi} \tag{0.11}$$

Furthermore

$$\delta_\gamma \phi = i\theta\phi \qquad \delta_\gamma \bar{\phi} = -i\theta\bar{\phi}\,. \tag{0.12}$$

We can write

$$\mathcal{L} = \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi + m^2|\phi|^2\,. \tag{0.13}$$

so that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\frac{\partial \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi}{\partial(\partial_\mu \phi)} = -\partial^\mu \bar{\phi} \tag{0.14}$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} = -\frac{\partial \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi}{\partial(\partial_\mu \bar{\phi})} = -\partial^\mu \phi \tag{0.15}$$

Hence we have all the ingredients in place to write down

$$j^\mu = -(\partial^\mu \bar{\phi})i\theta\phi - (\partial^\mu \phi)(-i\theta\bar{\phi}) = i\theta\left(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}\right) \tag{0.16}$$

Of course this is conserved for any θ, so that we might as well write this as

$$j^\mu = i\left(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}\right) \tag{0.17}$$

d) We work out

$$\bar{j}^\mu = -i\left(\phi\partial^\mu\bar{\phi} - \bar{\phi}\partial^\mu\phi\right) = j^\mu \tag{0.18}$$

so that the current must be real. A plane wave solution might be

$$\phi(\boldsymbol{x}) = \exp\left(ik_\mu x^\mu\right)\,. \tag{0.19}$$

Plugging this into the equations of motion gives

$$\begin{aligned} 0 &= (\partial_\mu\partial^\mu - m^2)\exp\left(ik_\nu x^\nu\right) = (\eta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\sigma} - m^2)\exp\left(ik_\nu x^\nu\right) \\ &= (-\eta^{\rho\sigma}k_\rho k_\sigma - m^2)\exp\left(ik^\nu x_\nu\right) = (-k_\nu k^\nu - m^2)\exp\left(ik^\nu x_\nu\right) \end{aligned} \tag{0.20}$$

so that we need to choose k_ν such that $k^\nu k_\nu = -k_0^2 + k_1^2 + k_2^2 + k_3^2 = -m^2$. Note that we could e.g. have solutions with $k_0 = \pm m$ and $k_i = 0$ for $i = 1, 2, 3$. We then have that

$$j^0 = -2k^0 \tag{0.21}$$

and depending on the sign of k^0 this is either positive or negative. If we wanted to use ϕ as a wave-function we would then be in trouble when trying to interpret j^0 as a probability density. This is why the Klein-Gordon equation could not be used to describe relativistic quantum mechanics using wave-functions.