Problem Class 4

Problem 1: Write down the most general real gauge invariant and Lorentz invariant Lagrangian with at most two derivatives for two complex scalar fields, ϕ of charge 1 and χ of charge 2, and a $U(1)$ gauge field A_μ which contains

1. standard kinetic terms for ϕ and χ;

   solution:

   Since the scalar fields are charged under a $U(1)$ gauge symmetry, we should replace the partial derivatives in the standard kinetic term $-\partial^\mu\bar{\phi}\partial_\mu\phi - \partial^\mu\bar{\chi}\partial_\mu\chi$ by gauge covariant derivatives. We need to remember that the gauge covariant derivative of a field of charge q is $D_\mu = \partial_\mu - iqA_\mu$. So we have the gauge invariant kinetic terms

   $$\mathcal{L}_{\text{kin}} = -(\partial^\mu\bar{\phi} + iA^\mu\bar{\phi})(\partial_\mu\phi - iA_\mu\phi) - (\partial^\mu\bar{\chi} + 2iA^\mu\bar{\chi})(\partial_\mu\chi - 2iA_\mu\chi)\ .$$

2. a kinetic term for A_μ;

   solution:

   This is simply the Maxwell Lagrangian density that we saw in the gauge theory formulation of electromagnetism:

   $$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu}$$

   where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

3. a real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates.

   solution:

   We start by noticing that for any fields f_1, f_2 of charges q_1, q_2 respectively under a $U(1)$ symmetry (global or local/gauge), their product f_1f_2 has charge $q_1 + q_2$. Indeed under a $U(1)$ transformation with group element $e^{i\alpha}$ the fields transform as

   $$(f_1, f_2) \mapsto (e^{iq_1\alpha}f_1, e^{iq_2\alpha}f_2) \qquad \Longrightarrow \qquad f_1f_2 \mapsto e^{i(q_1+q_2)\alpha}f_1f_2\ .$$

   This generalizes by induction to monomials in the fields: the charge of a monomial is the sum of the charges of its factors. In particular, a monomial in the fields is invariant under a $U(1)$ gauge transformation if and only if it has charge 0.

The scalar potential is a polynomial in $\phi, \chi, \bar{\phi}, \bar{\chi}$. Demanding gauge invariance (*i.e.* vanishing total charge), we see that the only allowed monomials are

$$1 \ , \quad |\phi|^2 = \bar{\phi}\phi \ , \quad |\chi|^2 = \bar{\chi}\chi \ , \quad \bar{\chi}\phi^2 \ , \quad \bar{\phi}^2\chi$$

and products/powers thereof. Therefore the most general real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates is

$$V(\bar{\phi}, \bar{\chi}, \phi, \chi) = V_0 + m_\phi^2|\phi|^2 + m_\chi^2|\chi|^2 + \mathrm{Re}(a\bar{\chi}\phi^2) + \lambda_\phi|\phi|^4 + \lambda_\chi|\chi|^4 + \lambda_{\phi\chi}|\phi|^2|\chi|^2 \ ,$$

where $V_0, m_\phi^2, m_\chi^2, \lambda_\phi, \lambda_\chi, \lambda_{\phi\chi}$ are real constants, and a is a complex constant. The constant V_0 (the 'vacuum energy density') is often ignored since it drops out of the equations of motion, and the energy is defined up to an additive constant.

Problem 2: Check by direct computation that

$$D_\mu\phi := (\partial - iA_\mu)\phi \rightarrow g(x)D_\mu\phi \tag{0.1}$$

for

$$\begin{aligned} \phi &\rightarrow g\phi \\ A_\mu &\rightarrow g(A_\mu + i\partial_\mu)g^{-1} \end{aligned} \tag{0.2}$$

solution: We work out

$$\begin{aligned} (\partial - iA_\mu)\phi &\rightarrow (\partial_\mu - ig(A_\mu g^{-1} + i\partial_\mu g^{-1}))g\phi \\ &= (\partial_\mu g)\phi + g\partial_\mu\phi - igA_\mu g^{-1}g\phi + (g\partial_\mu g^{-1})g\phi \\ &= gD_\mu\phi + (\partial_\mu g)\phi + (g\partial_\mu g^{-1})g\phi \end{aligned} \tag{0.3}$$

As $0 = \partial_\mu(gg^{-1}) = (\partial_\mu g)g^{-1} + g\partial_\mu g^{-1}$ the two last terms become

$$(\partial_\mu g)\phi - (\partial_\mu g)g^{-1}g\phi = 0 \tag{0.4}$$

and we can declare success!

Problem 3: Consider a gauge group G, with Lie algebra $\mathfrak{g}$.

1. Show by explicit calculation that a non-abelian gauge field configuration of the form

   $$A_\mu = ih(\partial_\mu h^{-1}) \ ,$$

   where $h(x)$ is a space-time dependent element of G, has field strength $F_{\mu\nu} = 0$.

   solution:

We calculate

$$\partial_\mu A_\nu = i(\partial_\mu h)(\partial_\nu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) \ ,$$

therefore

$$\begin{aligned}\partial_\mu A_\nu - \partial_\nu A_\mu &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) - ih(\partial_\nu \partial_\mu h^{-1}) \\ &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) \ ,\end{aligned}$$

where the second derivative terms cancel (as usual, we assume that h^{-1} is sufficiently differentiable so that Schwarz's/Clairaut's theorem applies). The contribution of the commutator is

$$\begin{aligned}-i[A_\mu, A_\nu] &= i[h\partial_\mu h^{-1}, h\partial_\nu h^{-1}] \\ &= ih(\partial_\mu h^{-1})h(\partial_\nu h^{-1}) - ih(\partial_\nu h^{-1})h(\partial_\mu h^{-1}) \ .\end{aligned}$$

Now we use the identity

$$0 = (\partial_\mu \mathbf{1}) = \partial_\mu(hh^{-1}) = (\partial_\mu h)h^{-1} + h(\partial_\mu h^{-1})$$

to get

$$\begin{aligned}-i[A_\mu, A_\nu] &= -i(\partial_\mu h)h^{-1}h(\partial_\nu h^{-1}) + i(\partial_\nu h)h^{-1}h(\partial_\mu h^{-1}) \\ &= -i(\partial_\mu h)(\partial_\nu h^{-1}) + i(\partial_\nu h)(\partial_\mu h^{-1}) \ .\end{aligned}$$

Hence

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0 \ .$$

2. Can you think of a simpler argument to reach the same conclusion?

solution:

Start from a configuration with vanishing gauge field $A_\mu = 0$. The field strength trivially vanishes: $F_{\mu\nu} = 0$. Now perform a gauge transformation with gauge parameter $g = h$. We find that the new (gauge transformed) gauge field A'_μ and field strength $F'_{\mu\nu}$ are

$$\begin{aligned}A'_\mu &= hA_\mu h^{-1} + ih(\partial_\mu h^{-1}) = ih(\partial_\mu h^{-1}) \\ F'_{\mu\nu} &= hF_{\mu\nu}h^{-1} = 0 \ .\end{aligned}$$

Now, what is primed or unprimed is a matter of point of view: I could have called the primed variables unprimed and vice versa, had I used the inverse gauge transformation. The key point here is that this shows that the field strength of $A_\mu = ih(\partial_\mu h^{-1})$ is $F_{\mu\nu} = 0$. Configurations like $A_\mu = ih(\partial_\mu h^{-1})$, which are obtained by a gauge transformation of the trivial (*i.e.* zero) configuration, are called *pure gauge* configurations.