27) (a) Express the Lagrangian density $\mathcal{L}_{\text{gauge}}$ in

$$S_{\text{gauge}}[A] = S_{YM}[A] + S_\theta[A] \ ,$$
$$\mathcal{L}_{\text{gauge}} = \mathcal{L}_{YM} + \mathcal{L}_\theta = -\frac{1}{2g_{YM}^2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{16\pi^2}\text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \ . \tag{1.1}$$

in terms of A^a_μ and the structure constants $f_{ab}{}^c$, and identify quadratic terms involving derivatives of the gauge field, and cubic and quartic terms in A_μ, which represent interactions.

(b) Show that the theta term

$$S_\theta[A] = \int d^4x \ \mathcal{L}_\theta \ ,$$
$$\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \ , \tag{1.2}$$

can be written as a surface (or 'boundary') term:

$$S_\theta = \frac{\theta}{8\pi^2}\int d^4x \ \partial_\mu K^\mu \ ,$$
$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\text{tr}(A_\nu\partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma) \ . \tag{1.3}$$

(c) Show that the equations of motion obtained from the action S_{gauge} are

$$D_\mu F^{\mu\nu} \equiv \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0 \ . \tag{1.4}$$

(d) Show, without using the EoM, that the Bianchi identity

$$D_\mu \tilde{F}^{\mu\nu} = 0 \ . \tag{1.5}$$

holds.

28) Consider the action

$$S[\phi,\bar{\phi},A] = S_{YM}[A] + S_\theta[A] + S_{\text{matter}}[\phi,\bar{\phi},A] \ .$$

(a) Show that the equations of motion are

$$D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi^\dagger}$$
$$D_\nu F^{\mu\nu} = g_{YM}^2 J^\mu \tag{1.6}$$

for a current $J_\mu = J^a_\mu t_a$ that you should find.

(b) Show that under a gauge transformation the current J^μ transforms as

$$J^\mu \mapsto g J^\mu g^{-1} \ , \tag{1.7}$$

and that J^μ is covariantly conserved, namely

$$D_\mu J^\mu = 0 \ . \tag{1.8}$$

29) Define

$$F^{U(1)}_{\mu\nu} := \frac{1}{2v} \mathrm{tr}(\Phi_\infty F_{\mu\nu}) \tag{1.9}$$

to be the field strength of the unbroken $H = U(1)$ subgroup of the gauge group $G = SU(2)$. Show that the magnetic charge

$$m^{U(1)} := \frac{1}{2\pi} \int_{S^2_\infty} \boldsymbol{B}^{U(1)} \cdot d^2\vec{\sigma} \tag{1.10}$$

of this unbroken $U(1)$ is proportional to the topological degree ν of Φ_∞, and find the proportionality factor.