1) Consider a Lorentz vector with components x^μ, which transforms under Lorentz transformations as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,.$$

Note that throughout this problem we are using summation convention.

a) Let $f^{\mu\nu} \equiv x^\mu x^\nu$. Find the transformation behavior of $f^{\mu\nu}$, $f^\mu{}_\nu = x^\mu x_\nu$ and $f_{\mu\nu} = x_\mu x_\nu$ under Lorentz transformations.

b) For another Lorentz vector y^μ, find the transformation behavior of $f^{\mu\nu} y_\mu$ under Lorentz transformations.

c) Compute

$$\sum_\mu \frac{\partial}{\partial x^\mu} x^\mu \,.$$

d) Work out the transformation behavior of

$$\frac{\partial}{\partial x^\mu}$$

under Lorentz transformations. Use c) to argue for the same result.

solution:

(a) We can infer the transformation of $f^{\mu\nu}, f^\mu{}_\nu, f_{\mu\nu}$ from that of x^μ and x_μ

$$\begin{aligned} f^{\mu\nu} &\to \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} f^{\mu'\nu'} \\ f^\mu{}_\nu &\to \Lambda^\mu{}_{\mu'} f^{\mu'}{}_{\nu'} (\Lambda^{-1})^{\nu'}_{\ \nu} \\ f_{\mu\nu} &\to f_{\mu'\nu'} (\Lambda^{-1})^{\mu'}_{\ \mu} (\Lambda^{-1})^{\nu'}_{\ \nu} \end{aligned} \tag{0.1}$$

Here the ordering of things I used on the rhs is not really important, I have written things in such a way that serves the slogan **upper indices transform with Λ and lower indices transform with Λ^{-1} from the right**.

(b) Using the result of a) and the fact that y_μ transforms with a Λ^{-1} we immediately see that

$$f^{\mu\nu} y_\mu \to \Lambda^\nu{}_{\nu'} f^{\mu\nu'} y_\mu \tag{0.2}$$

I.e. μ is a contracted dummy index and the only non-trivial transformation is coming from ν.

(c) This is simply

$$\sum_\mu \frac{\partial}{\partial x^\mu} x^\mu = \frac{\partial x^0}{\partial x^0} + \frac{\partial x^1}{\partial x^1} + \frac{\partial x^2}{\partial x^2} + \frac{\partial x^3}{\partial x^3} = 4 \tag{0.3}$$

(d) Writing

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{0.4}$$

implies that

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu}(\Lambda^{-1})^\nu{}_\mu \tag{0.5}$$

The derivative witht respect to a Lorentz vector hence transforms like a Lorentz covector. For this reason people usually write $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$. This result can also be seen from part c): The number 4 is a Lorentz scalar and as x^μ is a Lorentz vector $\frac{\partial}{\partial x^\mu}$ must be a covector to get something invariant.

2) Write a 4-vector (x^0, x^1, x^2, x^3) as a matrix M_x with $M_x^\dagger = M_x$:

$$M_x := \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} . \tag{0.6}$$

For $g \in SL(2,\mathbb{C})$ define an action $F(g)$ on $\mathbb{R}^4$ by

$$g \to F(g) \qquad F(g)M_x := gM_x g^\dagger . \tag{0.7}$$

a) Show that F is a homomorphism from $SL(2,\mathbb{C})$ to L.

b) For a rotation in the x^1, x^2-plane, find the element $g \in SL(2,\mathbb{C})$ that is mapped to it by F. Repeat the same for a boost along the x^1 direction.

solution:

a) First note that M_x is the most general 2×2 matrix with the property $M_x = M_x^\dagger$. This property is preserved by $F(g)$ as

$$(gM_x g^\dagger)^\dagger = g^{\dagger\dagger} M_x^\dagger g^\dagger = gM_x g^\dagger . \tag{0.8}$$

Furthermore $F(g)$ acts as a linear map on $\mathbb{R}^{1,3}$ which preserves $\det M_x$:

$$\det M_x \to \det(gM_x g^\dagger) = \det g \det M_x \det g^\dagger = \det M_x . \tag{0.9}$$

As $x_\mu x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\det M_x$, the linear map $F(g)$ preserves the length of vectors in $\mathbb{R}^{1,3}$ and is hence in L.

In fact, we can argue that $F(g)$ is contained in $L_+^\uparrow$ (you weren't asked this for the assignment, but it is good to know). The group $SL(2,\mathbb{C})$ is connected as the following argument shows: by a standard result from linear algebra, we can write any matrix in $SL(2,\mathbb{C})$ as

$$g = B \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} B^{-1} \tag{0.10}$$

We can now simply let b go to zero and a go to 1 continuously to connect any element in $SL(2,\mathbb{C})$ continuously to the identity. As $SL(2,\mathbb{C})$ is connected and L has four connected components, F can only map to one of them (it is a continuous map). Using $g = \mathbb{1}$ we see that F maps to $L_+^\uparrow$, the component containing the identity. This map is not injective, as g and $-g$ are mapped to the same $F(g)$.

b) Now let us consider how different matrices in $SL(2,\mathbb{C})$ act on M_x. First we investigate elements of $SL(2,\mathbb{C})$ that are in $SU(2)$. For $\theta \in \mathbb{R}$ set

$$\begin{aligned} g_3(\theta) &:= e^{i\theta\sigma_3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\ g_2(\theta) &:= e^{i\theta\sigma_2} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\ g_1(\theta) &:= e^{i\theta\sigma_1} = \begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix} \end{aligned} \tag{0.11}$$

Their action on M_x is

$$\begin{aligned} g_3 M_x g_3^\dagger &= \begin{pmatrix} x^0 + x^3 & e^{2i\theta}(x^1 - ix^2) \\ e^{-2i\theta}(x^1 + ix^2) & x^0 - x^3 \end{pmatrix} \\ g_2 M_x g_2^\dagger &= \begin{pmatrix} x^0 + x^1\sin(2\theta) + x^3\cos(2\theta) & x^1\cos(2\theta) - ix^2 - x^3\sin(2\theta) \\ x^1\cos(2\theta) + ix^2 - x^3\sin(2\theta) & x^0 - x^1\sin(2\theta) - x^3\cos(2\theta) \end{pmatrix} \\ g_1 M_x g_1^\dagger &= \begin{pmatrix} x^0 - x^2\sin(2\theta) + x^3\cos(2\theta) & x^1 - ix^2\cos(2\theta) - ix^3\sin(2\theta) \\ x^1 + ix^2\cos(2\theta) + ix^3\sin(2\theta) & x^0 + x^2\sin(2\theta) - x^3\cos(2\theta) \end{pmatrix} \end{aligned} \tag{0.12}$$

This is the same we observed when we studied the same action of $SU(2)$ on $\mathbb{R}^3$: g_i defines a rotation by angle of magnitude 2θ around the x_i axis is $\mathbb{R}^3$ with coordinates x^1, x^2, x^3. Similarly, one can parametrize rotations around arbitrary axis. As we can write any matrix in $SO(3)$ as product of such elementary rotations (see again problem class 1), the map from $SU(2) \subset SL(2,\mathbb{C})$ is surjective onto $SO(3) \subset L_+^\uparrow$.

We can realize three other independent elements of $SL(2,\mathbb{C})$ as

$$\begin{aligned} h_3(\theta) &:= e^{\theta\sigma_3} = \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \\ h_2(\theta) &:= e^{\theta\sigma_2} = \begin{pmatrix} \cosh(\theta) & -i\sinh(\theta) \\ i\sinh(\theta) & \cosh(\theta) \end{pmatrix} \\ h_1(\theta) &:= e^{\theta\sigma_1} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \end{aligned} \tag{0.13}$$

where again $\theta \in \mathbb{R}$. Note that now $h_j^\dagger \neq h_j^{-1}$ but still $\det h_j = 1$. We now have

$$h_3 M_x h_3^\dagger = \begin{pmatrix} e^{2\theta}(x^0+x^3) & x^1 - ix^2 \\ x^1 + ix^2 & e^{-2\theta}(x^0 - x^3) \end{pmatrix} \tag{0.14}$$

As $e^\theta = \cosh\theta + \sinh\theta$ for real θ we can summarize this as

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \to \begin{pmatrix} \cosh 2\theta & 0 & 0 & \sinh 2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh 2\theta & 0 & 0 & \cosh 2\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{0.15}$$

i.e. this exactly a boost in (minus) the x^3 direction. Boosts along the x^2 and x^1 axis are likewise realized by h_2 and h_1 and we can again find a boost along an arbitrary direction by exponentiating appropriate real linear combinations of the σ_j. In particular taking

$$h_1 = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{0.16}$$

gives

$$h_1 M h_1^\dagger = \begin{pmatrix} x_3 + x_0\cosh 2\theta + x_1 \sinh 2\theta & -ix_2 + x_1\cosh 2\theta + x_2\sinh 2\theta \\ ix_2 + x_1 \cosh 2\theta + x_2 \sinh 2\theta & -x_3 + x_0 \cosh 2\theta + x_1 \sinh 2\theta \end{pmatrix} \tag{0.17}$$

which is a boost in the x_1 direction.

You weren't asked to do this for the assignment, but we can now comment on how we should show that $F(g)$ is a surjective homomorphism. Given the above, it should be clear that we can write any rotation and any boost in $L_+^\uparrow$ as the image of an element of $SL(2,\mathbb{C})$ under $F(g)$. In the lectures we have stated a thoerem that every element of $L_+^\uparrow$ can be written as the product of an element of $SO(3)$ and a boost, hence every element in $L_+^\uparrow$ is in the image of F.

Here are some things to ponder:

1. How is the Lorent group defined? Why is it defined that way?
2. What's the point about upper/lower indices?