3) Verify that

$$l^{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad l^{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad l^{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$l^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad l^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad l^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{0.1}$$

are in the Lie algebra of L.

solution:

There are essentially two approaches here: a) finding the general form of Lie algebra elements of the Lie algebra $\mathfrak{so}(1,3)$ of the Lorentz group, and verifying that the above have this form, or b) finding paths in $L_+^\uparrow$ which give rise to these Lie algebra elements.

Let us first do approach a):
We have

$$\eta \Lambda^T \eta = \Lambda^{-1}$$

which after writing $\Lambda = e^\ell$ gives

$$\eta e^{\ell^T} \eta = e^{-\ell}\,.$$

This needs to hold for any real rescaling of ℓ as well, after all the set of Lie algebra elements ℓ form a vector space. We can hence write

$$\eta e^{t\ell^T} \eta = e^{-t\ell}\,.$$

for $t \in \mathbb{R}$. Taking a derivative w.r.t to t and setting $t = 0$ (which is equivalent to expanding to linear order) gives us

$$\eta \ell^T \eta = -\ell\,.$$

This says that $\ell^\mu{}_\mu = -\ell^\mu{}_\mu$ (no summation) so the diagonal is zero. For off-diagonal terms we have $\ell^i{}_j = -\ell^j{}_i$ for $i,j = 1,2,3$, $i \neq j$ and $\ell^0{}_i = -\ell^i{}_0$ for $i = 1,2,3$. The six matrices above are a basis of the vector space which is defined by these conditions.

Let us now do b):
As discussed in the lectures, rotations are in L, e.g. here is a path in L:

$$g^{12}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.2}$$

which gives

$$\ell^{12} = \frac{\partial}{\partial t} g^{12}(t)\Big|_{t=0} \tag{0.3}$$

and similarly for ℓ^{23} and ℓ^{13}. Then we have boosts, where we might look at

$$g^{01}(t) = \begin{pmatrix} \cosh t & -\sinh t & 0 & 0 \\ -\sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.4}$$

which gives

$$\ell^{01} = \frac{\partial}{\partial t} g^{01}(t)\Big|_{t=0} \tag{0.5}$$

and again similar for ℓ^{02} and ℓ^{03}.

4) The Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ -\mathbb{1}_{2\times 2} & 0 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad i = 1, 2, 3 \tag{0.6}$$

where $\mathbb{1}_{2\times 2}$ is the 2×2 identity matrix and σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{0.7}$$

a) Show that the Dirac matrices obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times 4}$.

b) Show the ‘freshers dream’:

$$(a_\mu \gamma^\mu)^2 = a_\mu a^\mu \mathbb{1}_{4\times 4} \tag{0.8}$$

solution:

(a) First note that $(\gamma^0)^2 = -\mathbb{1}_{4\times 4}$ and $(\gamma^i)^2 = \mathbb{1}_{4\times 4}$ (which follows from $(\sigma_i)^2 = \mathbb{1}_{2\times 2}$). Now we work out (for $i \neq j$)

$$\begin{aligned}\{\gamma^0,\gamma^i\} &= \begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} + \begin{pmatrix}-\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix} = 0\\ \{\gamma^i,\gamma^j\} &= \begin{pmatrix}\{\sigma_i,\sigma_j\} & 0\\ 0 & \{\sigma_i,\sigma_j\}\end{pmatrix} = 0\end{aligned} \tag{0.9}$$

(b) The reason this is called the 'freshers dream' is that it seems to say that $(\sum a_i)^2 = \sum a_i^2$ which is of course wrong. Using the Dirac matrices we can get something similar though. Let us first rewrite

$$\begin{aligned}(a_\mu\gamma^\mu)^2 = (a_\mu\gamma^\mu)(a_\nu\gamma^\nu) = a_\mu\gamma^\mu a_\nu\gamma^\nu &= \tfrac{1}{2}\left(a_\mu a_\nu\gamma^\mu\gamma^\nu + a_\mu a_\nu\gamma^\mu\gamma^\nu\right)\\ &= \tfrac{1}{2}\left(a_\mu a_\nu\gamma^\mu\gamma^\nu + a_\nu a_\mu\gamma^\nu\gamma^\mu\right)\end{aligned} \tag{0.10}$$

where we simply relabelled $\mu \leftrightarrow \nu$ in the second term. We can now write

$$(a_\mu\gamma^\mu)^2 = \tfrac{1}{2}a_\mu a_\nu\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = \tfrac{1}{2}a_\mu a_\nu 2\eta^{\mu\nu} = a^\mu a_\mu \tag{0.11}$$

It is equally fine to observe that all of the cross-terms cancel due to the Clifford algebra relation of the Dirac matrices.

5) Using the Dirac matrices, show that $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]$ are equal to

$$S^{0i} = \tfrac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}\,, \qquad S^{jk} = \frac{i}{2}\epsilon_{jkl}\begin{pmatrix}\sigma_l & 0\\ 0 & \sigma_l\end{pmatrix} \tag{0.12}$$

where i,j,k only take values $1,2,3$. What does this imply about the reducibility of the representation of $SL(2,\mathbb{C})$ defined by exponentiating the $S^{\mu\nu}$?

solution:

Using that $\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu$ when $\mu \neq \nu$ we have

$$S^{0i} = \tfrac{1}{4}[\gamma^0,\gamma^i] = \tfrac{1}{2}\gamma^0\gamma^i = \tfrac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix} \tag{0.13}$$

Furthermore

$$S^{jk} = \tfrac{1}{2}\gamma^j\gamma^k = \tfrac{1}{2}\begin{pmatrix}\sigma_j\sigma_k & 0\\ 0 & \sigma_j\sigma_k\end{pmatrix} \tag{0.14}$$

Now observe that (using $\sigma_j\sigma_k = -\sigma_k\sigma_j$ when $k \neq j$)

$$\sigma_j\sigma_k = \tfrac{1}{2}(\sigma_j\sigma_k + \sigma_j\sigma_k) = \tfrac{1}{2}(\sigma_j\sigma_k - \sigma_k\sigma_j) = \tfrac{1}{2}[\sigma_j,\sigma_k] = i\epsilon_{jkl}\sigma_l\,. \tag{0.15}$$

Hence

$$S^{jk} = \frac{i}{2}\epsilon_{jkl}\begin{pmatrix}\sigma_l & 0\\ 0 & \sigma_l\end{pmatrix} . \tag{0.16}$$

As discussed in the lectures, all of these are block diagonal, so this is a reducible representation. It is a representation of $SL(2,\mathbb{C})$ as we are effectively exponentiating complex linear combinations of Pauli matrices, which form the Lie algebra of $SL(2,\mathbb{C})$ as shown in Problem class 4 of Michaelmas term.

Here are some things to ponder:

1. What is the global structure of the Lorentz group?
2. How can we construct a representation of the Lie algebra of L using the Dirac matrices?