6) Verify that

   a) For an element $\Lambda(\theta) = e^{l^{12}\theta}$ of the Lorentz group (l^{12} is one of the generators of the Lorentz algebra introduced in the lectures) show that $\Lambda(0) = \Lambda(2\pi) = \mathbb{1}$. Now compare this behavior to the corresponding element of the representation acting on a Dirac spinor: $\Lambda_{1/2}(\theta) = e^{S^{12}\theta}$.

   b) Let $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. What is $\frac{1}{2}\left(\gamma^5 \pm \mathbb{1}\right)\Psi$ for Ψ a Dirac spinor written in terms of Weyl spinors?

solution:

(a) We compute

$$\Lambda(\theta) = e^{l^{12}\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.1}$$

which shows what we wanted to see. Now consider

$$\Lambda_{1/2}(\theta) = e^{S^{12}\theta} = \exp\left(\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\theta\right) = \begin{pmatrix} f(\theta) & 0 & 0 & 0 \\ 0 & \bar{f}(\theta) & 0 & 0 \\ 0 & 0 & f(\theta) & 0 \\ 0 & 0 & 0 & \bar{f}(\theta) \end{pmatrix} \tag{0.2}$$

where $f(\theta) = \cos(\theta/2) + i\sin(\theta/2)$. Hence $\Lambda_{1/2}(0) = -\Lambda_{1/2}(2\pi) = \Lambda_{1/2}(4\pi)$, similar as for the spinors in $\mathbb{R}^3$ as we observed before.

(b) We compute

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{0.3}$$

For a Dirac spinor Ψ written in terms of Weyl spinors this maps $\psi_{L/R}$ to $\mp\psi_{L/R}$. We can use this to project to $\psi_{L/R}$:

$$\tfrac{1}{2}\left(\gamma^5 + \mathbb{1}\right)\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \tag{0.4}$$

and

$$\tfrac{1}{2}\left(\gamma^5 - \mathbb{1}\right)\Psi = -\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \tag{0.5}$$

7) How does

$$B^{\mu\nu} \equiv \bar{\Psi}\gamma^\mu\gamma^\nu\Psi$$

transform under Lorentz transformations for Ψ a Dirac spinor?

solution: We can work this out using the same logic used in the lectures.

$$\begin{aligned} B^{\mu\nu} \to &\Psi^* \Lambda_{1/2}^\dagger \gamma_0 \gamma^\mu \gamma^\nu \Lambda_{1/2} \Psi = \Psi^* \gamma_0 \Lambda_{1/2}^{-1} \gamma^\mu \gamma^\nu \Lambda_{1/2} \Psi \\ =&\Psi^* \gamma_0 \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} \Lambda_{1/2}^{-1} \gamma^\nu \Lambda_{1/2} \Psi \\ =&\Psi^* \gamma_0 \Lambda^\mu{}_{\mu'} \gamma^{\mu'} \Lambda^\nu{}_{\nu'} \gamma^{\nu'} \Lambda_{1/2} \Psi \\ =&\Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} B^{\mu'\nu'} \end{aligned} \tag{0.6}$$

Hence $B^{\mu\nu}$ transforms as we would expect given its indices!

8) For a Dirac spinor Ψ write

$$\bar{\Psi}\gamma^\mu\Psi \qquad \text{and} \qquad \bar{\Psi}\Psi$$

in terms of Weyl spinors.

solution: All we need to do is unpack the above expression and write $\Psi = (\psi_L, \psi_R)$. For $\mu = 0$ we find

$$\bar{\Psi}\gamma^\mu\Psi = \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} (\gamma^0)^2 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -|\psi_L|^2 - |\psi_R|^2 \tag{0.7}$$

while for $\mu = i$ we find

$$\bar{\Psi}\gamma^\mu\Psi = \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} \gamma^0\gamma^i \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \psi_L^*\sigma_i\psi_L - \psi_R^*\sigma_i\psi_R \tag{0.8}$$

using the expressions for Dirac matrices in terms of Pauli matrices.

9) For a relativistic point particle moving on path C through space-time, the only Lorentz invariant property of C is its length. Taking the action of a relativistic particle to be the length of C and parametrizing C as $x^\mu(s)$ we can write this as

$$S[x^\mu, \dot{x}^\mu] = -cm \int_C ds = -cm \int_C \sqrt{-\dot{x}^\mu \dot{x}_\mu} ds \,. \tag{0.9}$$

for a constant m and c the speed of light and $\dot{x}^\mu = \partial/\partial s \; x^\mu$. C is called the world-line of the particle.

a) Show that this action is invariant under Lorentz transformations.

b) Find the equations of motions and show that they are solved by straight lines in space-time.

c) Set $s = t$ and expand the action for slow particles to recover the action of a non-relativistic point particle.

solution:

a) Under a Lorentz transformation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{0.10}$$

and so

$$\dot{x}^\mu \to \Lambda^\mu{}_\nu \dot{x}^\nu \tag{0.11}$$

By definition

$$\dot{x}^\mu \dot{x}_\mu \tag{0.12}$$

is invariant under Lorentz transformations.

b) The Euler Lagrange eqs are

$$\frac{d}{ds}\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} = 0 \tag{0.13}$$

Straight lines are given by e.g. $x^\mu(s) = s\, c^\mu + x_0^\mu$, i.e. $\dot{x}^\mu(s) = c^\mu$ for c^μ constants such that $c^\mu c_\mu < 0$ ('time-like curves'). As this makes

$$\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} \tag{0.14}$$

constant as a function of s they obey the equations of motion.

c) We can set $s = t$ and expand for small speeds to find

$$L = cm\sqrt{-\dot{x}^\mu \dot{x}_\mu} = cm\sqrt{c^2 - \boldsymbol{v}^2} \sim mc^2 - \frac{1}{2}m\boldsymbol{v}^2\,. \tag{0.15}$$

Up to a sign, this is the usual expression for the kinetic energy of a point particle with a constant 'potential' term mc^2. This is the origin of the famous mass-energy relation $E = mc^2$.

Remark: A more elegant treatment starts with the observation that we can reparametise the action by sending $s \to s(u)$ such that $-\dot{x}^\mu \dot{x}_\mu = 1$ for all u. This simplifies all formulas, shows that straight lines are the only solutions and identifies u as the proper time of a observer travelling along $x^\mu(s)$.

Here are some things to ponder:

1. In which ways is the relationship between $SO(3)$ and $SU(2)$ the same as the relationship between the Lorentz group and $SL(2,\mathbb{C})$.
2. What is a spinor?
3. What is an action?