10) Consider the following action of a real scalar field

$$S = \int d^4x\, \partial_\mu\phi\partial^\mu\phi + m^2\phi^2\,.$$

Show that the equations of motion are

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0\,.$$

solution:

We have

$$\partial\mathcal{L}/\partial\phi = 2m^2\phi \tag{0.1}$$

and

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu\phi)}\mathcal{L} &= \frac{\partial}{\partial(\partial_\mu\phi)}\,\partial_\rho\phi\partial^\rho\phi = \eta^{\rho\sigma}\frac{\partial}{\partial(\partial_\mu\phi)}\partial_\rho\phi\partial_\sigma\phi \\ &= \eta^{\rho\sigma}\partial_\sigma\phi\frac{\partial}{\partial(\partial_\mu\phi)}\,\partial_\rho\phi + \eta^{\rho\sigma}\partial_\rho\phi\frac{\partial}{\partial(\partial_\mu\phi)}\,\partial_\sigma\phi \\ &= \eta^{\rho\sigma}\delta^\mu{}_\rho\partial_\sigma\phi + \eta^{\rho\sigma}\delta^\mu{}_\sigma\partial_\rho\phi = 2\partial^\mu\phi\end{aligned} \tag{0.2}$$

The equation of motion for ϕ is hence

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{0.3}$$

Note that we can write the Lagrangian density as

$$\mathcal{L} = -(\frac{\partial}{\partial t}\phi)^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 \tag{0.4}$$

so this is really the same as example 4.2. You can check that the equations of motion are also the same in both cases.

11) Consider the action

$$S = \int d^4x\bar{\Psi}\left(\gamma^\mu\partial_\mu + m\right)\Psi\,.$$

for a Dirac spinor Ψ.

a) Show that S is Lorentz invariant.

b) Find the equations of motion.

c) Find the conserved charge associated to the $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$.

d) Show that

$$\left(\gamma^\mu\partial_\mu - m\right)\left(\gamma^\nu\partial_\nu + m\right) = \partial_\mu\partial^\mu - m^2$$

solution:

(a) We have already seen the transformation behavior of all of the terms in this action when we replace ∂_μ by a constant Lorentz covector a_μ in the third problem class, where we found that they are all invariant. Transforming the argument of the spinor field Ψ effectively makes ∂_μ transform as a Lorentz covector as well, so that the above action is Lorentz invariant.

Let's translate the above into equations. We let $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ and $\boldsymbol{y} = \Lambda^{-1}\boldsymbol{x}$, so that a Lorentz transformation maps

$$\Psi(\boldsymbol{x}) \to \Lambda_{1/2}\Psi(\boldsymbol{y})\,. \tag{0.5}$$

Note that $\Lambda_{1/2}$ is the 'spinor representation' matrix associated to Λ, i.e. if $\Lambda = e^{l^{\rho\sigma}\theta_{\rho\sigma}}$ then $\Lambda_{1/2} = e^{S^{\rho\sigma}\theta_{\rho\sigma}}$.

Using the transformation of $\bar{\Psi}$ studied before, $\bar{\Psi} \to \bar{\Psi}\Lambda_{1/2}^{-1}$ we find

$$\begin{aligned} S \to S' &= \int d^4x \bar{\Psi}(\boldsymbol{y})\Lambda_{1/2}^{-1}\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right)\Lambda_{1/2}\Psi(\boldsymbol{y}) \\ &= \int d^4y \bar{\Psi}(\boldsymbol{y})\Lambda_{1/2}^{-1}\left(\gamma^\mu \left(\Lambda^{-1}\right)^\rho{}_\mu \frac{\partial}{\partial y^\rho} + m\right)\Lambda_{1/2}\Psi(\boldsymbol{y}) \end{aligned}$$

where we have used the fact that the dervative behaves like a covector (via the product rule) and that $d^4x = d^4y$ for proper Lorentz transformations. Now we use the magical formula $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$. We then have

$$S' = \int d^4y \bar{\Psi}(\boldsymbol{y})\left(\gamma^\nu \left(\Lambda^{-1}\right)^\rho{}_\mu \Lambda^\mu{}_\nu \frac{\partial}{\partial y^\rho} + \Lambda_{1/2}^{-1}\Lambda_{1/2} m\right)\Psi(\boldsymbol{y}) \tag{0.6}$$

I have rearranged some factors (which are just numbers as we are using indices) and you can see that $(\Lambda^{-1})^\rho{}_\mu\Lambda^\mu{}_\nu = \delta^\rho{}_\nu$. As also $\Lambda_{1/2}^{-1}\Lambda_{1/2} = \mathbb{1}$ we end up with

$$S' = \int d^4y \bar{\Psi}(\boldsymbol{y})\left(\gamma^\nu \frac{\partial}{\partial y^\nu} + m\right)\Psi(\boldsymbol{y}) = S\,. \tag{0.7}$$

as it is now evident that all that has happened is that $\boldsymbol{x}$ has been relabelled as $\boldsymbol{y}$ everywhere.

(b) To find the field equation for $\bar{\Psi}$, let us write out the Lagrangian in terms of the components Ψ_I of the spinors:

$$\mathcal{L} = \Psi_I^*\gamma^0_{IJ}\left(\gamma^\mu_{JK}\partial_\mu + \delta_{JK}m\right)\Psi_K \tag{0.8}$$

where γ^0_{IJ} and γ^μ_{JK} are the components of these matrices. The Euler-Lagrange equation for Ψ^* is simply

$$\frac{\partial \mathcal{L}}{\partial \Psi^*_I} = 0 \tag{0.9}$$

as there are no derivatives w.r.t Ψ^* in $\mathcal{L}$. We hence find

$$\gamma^0_{IJ} \left(\gamma^\mu_{JK} \partial_\mu + \delta_{JK} m\right) \Psi_K = 0\,. \tag{0.10}$$

Multiplying by $(\gamma^0)^{-1}$ gives

$$\left(\gamma^\mu \partial_\mu + m\right) \Psi = 0\,. \tag{0.11}$$

This is the celebrated Dirac equation.

(c) Under the $U(1)$ symmetry acting on Ψ as $\Psi \to e^{i\theta}\Psi$, or in components $\Psi_I \to e^{i\theta}\Psi_I$. Ψ has 4 components Ψ_I, each of which is complex, so we need to treat Ψ_I and $\bar{\Psi}_I$ as 8 independent fields. The infinitesimal transformation are found by expanding to linear order in θ:

$$\delta\Psi_I = i\theta\Psi_I \qquad \delta\Psi_I = -i\theta\Psi_I \tag{0.12}$$

and the conserved current is (note we are using summation convention below, i.e. summing over I)

$$\begin{aligned} j^\mu &= \delta\Psi_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_I)} + \delta\bar{\Psi}_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_I)} \\ &= i\theta\Psi_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_I)} - i\theta\bar{\Psi}_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_I)} \\ &= i\theta\Psi_I \Psi^*_K \gamma^0_{KJ} \gamma^\mu_{JI} = i\theta\bar{\Psi}\gamma^\mu\Psi\,. \end{aligned} \tag{0.13}$$

Again this is conserved for any θ, and we don't loose anything rescaling the current to get rid of the $i\theta$ in the factor.

Rescaling this we get the conserved charge density $j^0 = -\bar{\Psi}\gamma^0\gamma^0\Psi = \bar{\Psi}\Psi$, i.e. the conserved charge is

$$Q_V = \int_V d^3x\, |\Psi|^2 \tag{0.14}$$

which is positive definite. Hence one can use Ψ as a wave-function just as one does for the Schroedinger equation.

(d) We work out

$$\begin{aligned}(\gamma^\mu\partial_\mu - m)(\gamma^\nu\partial_\nu + m) &= \left(\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2\right)\\ &= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2\right)\\ &= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\nu\partial_\mu\gamma^\mu\gamma^\nu - m^2\right)\\ &= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\nu\gamma^\mu - m^2\right)\\ &= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} - m^2\right) = \left(\partial_\mu\partial_\nu\eta^{\mu\nu} - m^2\right)\\ &= \left(\partial_\mu\partial^\mu - m^2\right)\end{aligned} \tag{0.15}$$

Note that we have simply relabelled μ and ν for the second term in the 4th line. The same result can be found by writing out the sums $\gamma^\mu\partial_\mu$ and $\gamma^\nu\partial_\nu$ and collecting all the terms. It is in the sense of the above equation that the Dirac equation is the square root of the Klein-Gordon equation. The above computation is that prompted Dirac to introduce the Dirac matrices.

12) Consider a field Φ transforming in the adjoint representation of the Lie group $SU(n)$. Show that

$$S = \int d^4x \ \mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right)$$

is invariant under the action of $SU(n)$ and find the associated conserved current.

solution:

We first need to think about what it means to transform in the adjoint representation. The adjoint representation acts on the vector space that is equal to the Lie algebra of $SU(n)$. We should hence think of Φ as a (space-time dependent) element of the Lie algebra of $SU(n)$. In particular, this means Φ is a traceless anti-hermitian $n\times n$ matrix that transforms as

$$\Phi \to g^{-1}\Phi g \tag{0.16}$$

for $g \in SU(n)$ and also

$$\partial_\mu\Phi \to g^{-1}(\partial_\mu\Phi)g \tag{0.17}$$

Under this map

$$\begin{aligned}&\mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right) \to \mathrm{tr}\left(g^{-1}\partial_\mu\Phi g g^{-1}\partial^\mu\Phi g\right)\\ &= \mathrm{tr}\left(g^{-1}\partial_\mu\Phi\partial^\mu\Phi g\right) = \mathrm{tr}\left(g g^{-1}\partial_\mu\Phi\partial^\mu\Phi\right) \ = \mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right)\end{aligned} \tag{0.18}$$

using the properties of the trace. The associated infinitesimal transformation (Lie algebra representation) is

$$\delta_\gamma \Phi = [\Phi, \gamma] \tag{0.19}$$

for $\gamma \in \mathfrak{su}(n)$. For a basis γ_i of the Lie algebra we can write

$$\Phi = \Phi_i \gamma_i \tag{0.20}$$

so that

$$\mathcal{L} = \partial_\mu \Phi_i \partial^\mu \Phi_j \mathrm{tr}\left(\gamma_i \gamma_j\right) \tag{0.21}$$

and

$$\gamma_i \delta_\gamma \Phi_i = \Phi_j [\gamma_j, \gamma] \tag{0.22}$$

We can now work out

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} \delta_\gamma \phi_i = 2(\partial^\mu \Phi_j)\mathrm{tr}(\gamma_i \gamma_j) \delta_\gamma \Phi_i = 2\mathrm{tr}\left(\delta_\gamma \Phi \partial^\mu \Phi\right) \\ &= 2\mathrm{tr}\left([\Phi, \gamma]\partial^\mu \Phi\right) \end{aligned} \tag{0.23}$$

Here are some things to ponder:

1. What is an action?
2. What is a symmetry of an action?
3. When do we consider a physical system to be Lorentz invariant?